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To Thomas Stewart Wesner

Dad

To Margot

Phil

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C H A P T E R 11

The Conic Sections

In this chapter we study more of analytic geometry, the area of mathematics that connects algebra and geometry. Recall that in chapter 3 we studied points, lines, and circles. We also studied parabolas in chapter 4.

In this chapter we study the parabola in more detail, and learn about the ellipse and the hyperbola.[1] The point, line, circle, parabola, ellipse, and hyperbola comprise a family of curves called the **conic sections.** This is because each curve can be found by slicing a right circular cone, as shown in figure 11–1. This fact was discovered by the Greek Menaechmus.[2]

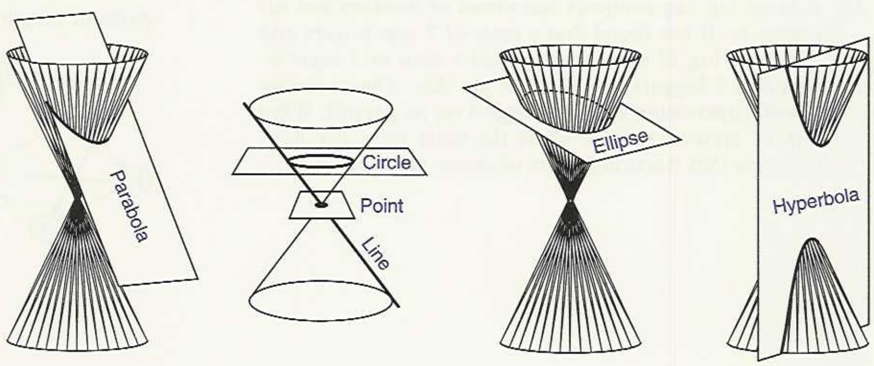

Figure 11–1

[1]The names parabola, ellipse, and hyperbola were created by the Greek Apollonius of Perga, (approximately 262 to 190 B.C.). They mean comparison, deficiency, and excess, respectively. These meanings come from the fact that these geometric figures were used by the Greeks in solving quadratic equations.

[2]Menaechmus was a student of Eudoxus, perhaps the best of the ancient Greek mathematicians. Eudoxus was in turn a student of the Platonic Academy in Athens, founded by Plato, best known for his work in philosophy. Plato has been called the "maker of mathematicians"; legend has it that above the doors of his school was inscribed the motto "Let no one ignorant of geometry enter here." This mathematical activity spanned approximately the years 400 B.C. to 300 B.C.

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11–1 The parabola

A stunt person is going to jump off a 40-foot-high building, running at an estimated velocity of 4 feet per second (horizontally). How far from the base of the building will this person land?

The parabola is the mathematical object that models this and many other situations. In section 4–1 we graphed parabolas. We now define a parabola from a geometric viewpoint. *A parabola is the set of all points equidistant*[3] *from a line and a point that is not on that line.* The line is called the **directrix,** and the point is called the **focus.** Figure 11–2 illustrates this.

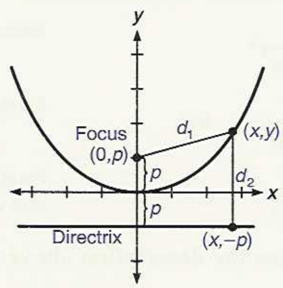


Figure 11–2

In figure 11–2 we have placed the focus at $(0,p)$, and the directrix is the line $y = -p$. Thus, $|p|$ equals half the distance from the focus to the directrix.

Let (x,y) be any point on the parabola (see the figure). Then we can proceed as follows:

$d_1 = \sqrt{x^2 + (y-p)^2}$	Apply the distance formula (section 3–1) to the points (x,y) and $(0,p)$
$d_2 = y + p$	Distance formula or examination of the figure
$d_1 = d_2$	Definition of parabola
$\sqrt{x^2 + (y-p)^2} = y + p$	Replace d_1 and d_2 by the values above
$x^2 + (y-p)^2 = (y+p)^2$	Square both members
$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$	Square the binomials
$x^2 = 4py$	Simplify the equation
$y = \dfrac{1}{4p}x^2$	Divide both members by $4p$, and interchange the members

Thus, we arrive at the analytic (algebraic) description of a parabola.

[3] Equidistant means "equal distance."

Parabola

$$y = \frac{1}{4p}x^2$$

is the equation of a parabola with vertex at (0,0), focus $(0,p)$, and directrix the line $y = -p$. The y-axis is an axis of symmetry.

The parabola opens up if $p > 0$ and down if $p < 0$. Note that the only intercept for an equation of this form is at the vertex, (0,0).

In section 4–1 we graphed equations of the form $y = ax^2 + bx + c$, $a \neq 0$, as a parabola, by completing the square, finding the vertex, and using linear transformations. Transformations can be applied as follows:

$$y = \frac{1}{4p}x^2$$ Basic parabola with vertex at (0,0)

$$y = \frac{1}{4p}(x - h)^2$$ Parabola translated horizontally h units

$$y = \frac{1}{4p}(x - h)^2 + k$$ Parabola translated horizontally h units and vertically k units

We can generalize the description above as follows:

Parabola (general)

$$y = \frac{1}{4p}(x - h)^2 + k$$

is the equation of a parabola with vertex at (h,k).

The focus is p units above or below the vertex, at $(h,k + p)$. The directrix is a horizontal line p units above or below the vertex, at $y = k - p$. The graph is symmetric (forms a mirror image) about the vertical line $x = h$. This line is called the **axis of symmetry.**

To graph a parabola of the form $y = \frac{1}{4p}(x - h)^2 + k$ using algebraic methods:

- The vertex is at (h,k).
- The focus is at $(h,k + p)$.
- The directrix is the line $y = k - p$.
- Compute the x-intercepts: set $y = 0$ and solve for x.
- Compute the y-intercept: set $x = 0$ and solve for y.
- The line $x = h$ is an axis of symmetry.

In general, to graph a parabola, we compute the focus, vertex, x- and y-intercepts, and directrix. We plot these values and the axis of symmetry. If we do not have enough points for a reasonably accurate graph we plot a few additional points.

Of course a modern graphing calculator can be used to draw the graph. (The steps for the TI-81 are shown throughout this chapter.) It is still useful to calculate the focus, vertex, x- and y-intercepts, and directrix.

Section 4–1 illustrated graphing parabolas. Example 11–1 A illustrates graphing a parabola using the information provided by the vertex and intercepts. It also illustrates finding the focus and directrix.

■ Example 11–1 A

Graph the parabola; clearly state the x- and y-intercepts, focus, directrix, and vertex.

$y = x^2 - 6x + 4$

$y = x^2 - 6x + 9 - 9 + 4$ — Complete the square (section 4–1)

$y = (x - 3)^2 - 5$

$y = (x - 3)^2 + (-5)$

Vertex: $(3,-5)$

p: $\dfrac{1}{4p} = 1$

$1 = 4p$ — Multiply each member by $4p$

$\frac{1}{4} = p$ — Divide each member by 4

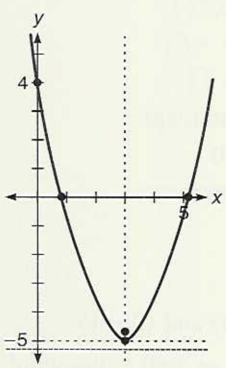
Focus: $(3,-5 + \frac{1}{4}) = (3,-4\frac{3}{4})$

Directrix: $y = -5 - \frac{1}{4} = -5\frac{1}{4}$

Axis of symmetry: The line $x = 3$.

y-intercept: Let $x = 0$.

$y = 0^2 - 6(0) + 4 = 4$; $(0,4)$



x-intercepts: Let $y = 0$.

$0 = (x - 3)^2 - 5$ — Use this form for x

$5 = (x - 3)^2$

$\pm\sqrt{5} = x - 3$ — Square root of both members

$3 \pm \sqrt{5} = x$ — Add 3 to both members

$x \approx 0.8, 5.2$; $(0.8,0)(5.2,0)$

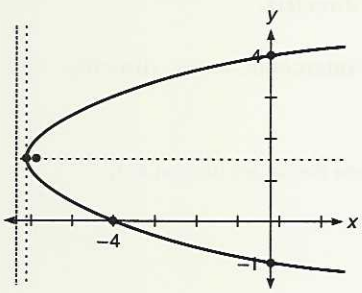
$\boxed{\text{Y=}}$ $\boxed{\text{X|T}}$ $\boxed{x^2}$ $\boxed{-}$ 6 $\boxed{\text{X|T}}$ $\boxed{+}$ 4 $\boxed{\text{RANGE } -1,6,-6,5}$ ■

All of the development to this point has focused on equations in which the x variable is squared. A parabola also results when it is the y variable that is squared (and the x variable is not squared). In this situation, the parabola opens to the right or left instead of up or down. Rather than try to learn a new technique for this situation, we can combine an old one with what we have just seen.

If we exchange the variables x and y in an equation, then obtain important points that are part of this new equation, we can obtain important points that relate to the original equation by reversing the values in the ordered pairs.

What we are doing when we do this is creating the inverse of a given relation, studying this inverse relation, and then applying what we discover to the original relation. (See section 4–5.) These ideas are illustrated in example 11–1 B.

■ *Example 11–1 B*



Graph the parabola $x = y^2 - 3y - 4$.

This parabola is quadratic in the variable y, not x. One way to graph it is to compute key points for its inverse, then reverse the points.

$x = y^2 - 3y - 4$

$y = x^2 - 3x - 4$ — Replace x by y and y by x; This is the inverse relation to the original relation

$y = x^2 - 3x + \frac{9}{4} - \frac{9}{4} - 4$ — Complete the square

$y = (x - \frac{3}{2})^2 - \frac{25}{4}$ — Vertex at $(\frac{3}{2}, -\frac{25}{4}) = (1\frac{1}{2}, -6\frac{1}{4})$

As we compute key points for this inverse relation we keep a record of the reverse of these points—these are the points in the original relation.

$y = x^2 - 3x - 4$	$x = y^2 - 3y - 4$
Vertex: $(1\frac{1}{2}, -6\frac{1}{4})$	$(-6\frac{1}{4}, 1\frac{1}{2})$
p: $\dfrac{1}{4p} = 1$	
$1 = 4p$	
$\frac{1}{4} = p$	
Focus: $(1\frac{1}{2}, -6\frac{1}{4} + \frac{1}{4}) = (1\frac{1}{2}, -6)$	$(-6, 1\frac{1}{2})$
Directrix: $y = -6\frac{1}{4} - \frac{1}{4} = -6\frac{1}{2}$	$x = -6\frac{1}{2}$
Axis of symmetry: $x = 1\frac{1}{2}$	$y = 1\frac{1}{2}$
y-intercept: Let $x = 0$.	x-intercept:
$y = 0^2 - 3(0) - 4 = 4$; $(0,4)$	$(4,0)$
x-intercepts: Let $y = 0$.	y-intercepts:
$0 = x^2 - 3x - 4$	
$0 = (x - 4)(x + 1)$	
$x - 4 = 0$ or $x + 1 = 0$	
$x = 4$ or $x = -1$; $(4,0)$ and $(-1,0)$	$(0,4)$ and $(0,-1)$

Plotting the vertex, focus, x-intercept,and y-intercepts, as well as knowing the axis of symmetry, and that a parabola in which the y variable is squared opens horizontally, allows us to draw the graph of the original relation.

A graphing calculator cannot be used directly to graph this parabola. This is because this relation is not a function. (Section 3–5 showed that any graph which fails the ''vertical line test'' is not a function.) Algebraically this means

that the equation can not be solved for y without obtaining at least two solutions. The following shows how to graph this equation with a graphing calculator.

$x = y^2 - 3y - 4$ First solve for y by completing the square (section 4–1)
$y^2 - 3y = x + 4$
$y^2 - 3y + (\frac{3}{2})^2 = x + 4 + (\frac{3}{2})^2$
$(y - \frac{3}{2})^2 = x + \frac{25}{4}$
$y - \frac{3}{2} = \pm\sqrt{x + \frac{25}{4}}$
$y = \pm\sqrt{x + \frac{25}{4}} + \frac{3}{2}$

Now graph each function $y = \sqrt{x + \frac{25}{4}} + \frac{3}{2}$ and $y = -\sqrt{x + \frac{25}{4}} + \frac{3}{2}$. Do this by entering:

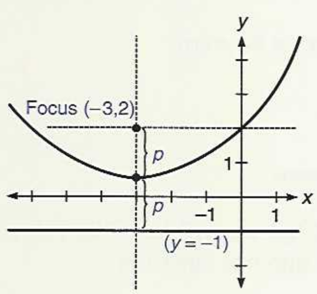
$\boxed{\text{Y=}}$ $\boxed{\sqrt{\ }}$ $\boxed{(}$ $\boxed{\text{X|T}}$ $\boxed{+}$ 6.25 $\boxed{)}$ $\quad Y_1 = \sqrt{x + 6.25}$
$\boxed{\text{ENTER}}$ $\boxed{\text{Y-VARS}}$ $\boxed{\text{ENTER}}$ $\boxed{+}$ 1.5 $\quad Y_2 = Y_1 + 1.5$
$\boxed{\text{ENTER}}$ $\boxed{(-)}$ $\boxed{\text{Y-VARS}}$ $\boxed{\text{ENTER}}$ $\boxed{+}$ 1.5 $\quad Y_3 = -Y_1 + 1.5$

Now use the cursor arrows to place the blinking cursor on the = in ":Y_1=√(X+6.25)" and hit $\boxed{\text{ENTER}}$. This turns off the graphing of Y_1. Now select $\boxed{\text{Range } -7,1,-2,5}$ $\boxed{\text{GRAPH}}$. ■

Example 11–1 C illustrates finding the equation of a parabola given some of its algebraic properties.

■ Example 11–1 C

Determine the equation of a parabola with focus at $(-3,2)$, opening vertically, and directrix at $y = -1$.



The x value of the vertex is -3, since it is directly below the focus at $(-3,2)$.

The vertex is half-way between the focus and directrix, so its y value must be half way "between" the y values 2 and -1. This value is their average: $\frac{2 + (-1)}{2} = \frac{1}{2}$.

Thus the vertex is at $(-3,\frac{1}{2})$.The distance between the focus and vertex is $|p|$, so $|p| = 1\frac{1}{2}$. Since the parabola opens upward $p > 0$, so $p = 1\frac{1}{2}$.

$y = \frac{1}{4p}(x - h)^2 + k$ Basic equation of a parabola with vertex at (h,k)

$y = \frac{1}{4(1\frac{1}{2})}(x + 3)^2 + \frac{1}{2}$ $p = 1\frac{1}{2}$, $(h,k) = (-3,\frac{1}{2})$

$y = \frac{1}{6}(x + 3)^2 + \frac{1}{2}$ $\frac{1}{4(1\frac{1}{2})} = \frac{1}{4(\frac{3}{2})} = \frac{1}{6}$

$6y = (x + 3)^2 + 3$ Multiply each member by 6

$6y = x^2 + 6x + 12$ $(x + 3)^2 + 3 = (x^2 + 6x + 9) + 3$

$y = \frac{1}{6}x^2 + x + 2$ Divide each member by 6 ■

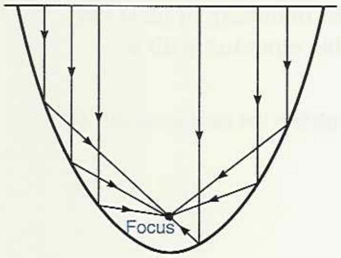


Figure 11–3

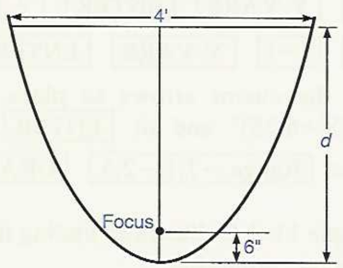
An important property of parabolas

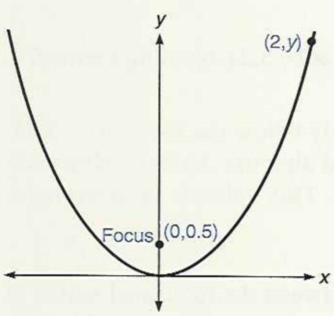
A parabola has the important property that if a mirror is shaped in the form of a parabola, then parallel light rays entering the reflector parallel to the axis of symmetry are reflected to the focus (see figure 11–3). This is the principle behind dish antennas used to receive TV signals from space satellites. The same principle, in reverse, is why the reflecting portion of a flashlight or automobile headlight is parabolic.

Example 11–1 D illustrates how to find the equation of a parabola given certain information about its dimensions.

■ ***Example 11–1 D***

The width of the parabolic mirror determines how much light or radio waves are collected. Suppose a parabolic mirror is constructed so that the focus is 6″ from the vertex. The mirror is to be 4′ across. Determine the height of the mirror, d in the figure.





If we put this figure in a coordinate system as shown, we have the vertex at (0,0) and $p = \frac{1}{2}$ ft.

$$y = \frac{1}{4p}x^2 \quad \text{Basic parabola with vertex at the origin}$$

$$y = \frac{1}{4(\frac{1}{2})}x^2 \quad p = \frac{1}{2}$$

$$y = \frac{1}{2}x^2 \quad \text{The equation of the antenna}$$

If we find the value of y shown in the coordinate $(2,y)$ we will have the value of d we desire. For this we insert the value $x = 2$ into our equation.

$$y = \frac{1}{2}(2^2) \quad \text{Find } y \text{ in } (2,y) \text{ by replacing } x \text{ by 2 in the equation}$$

$$y = 2 \quad \text{The distance } d \text{ is 2 feet}$$

■

The path of a falling object

Near the surface of the earth an object with no initial vertical velocity falls a distance $d = 16t^2$, where t is time in seconds. For instance in $\frac{1}{2}$ second an object will fall $d = 16(\frac{1}{2})^2 = 4$ feet. If the object has a constant horizontal velocity v, its trajectory will be a parabola. This can be seen as follows (see

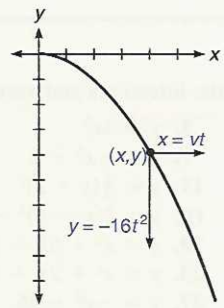


Figure 11–4

figure 11–4). The horizontal distance covered is the product of rate and time, so it is $x = vt$ (v in feet per second and t in seconds), and the vertical distance is $y = -16t^2$ (we use the negative value to indicate the object is falling). Thus,

Horizontal motion	Vertical motion
$x = vt$	
$x^2 = v^2t^2$	$y = -16t^2$
$\frac{x^2}{v^2} = t^2$	$-\frac{y}{16} = t^2$

If we replace t^2 in the horizontal motion equation by $-\frac{y}{16}$ we obtain $\frac{x^2}{v^2} = -\frac{y}{16}$, so $y = -\frac{16}{v^2}x^2$. (This is an example of the method of substitution for expression as illustrated in section 3–2.) For a given value of v this equation is a parabola. Example 11–1 E illustrates a use of this information.

■ ***Example 11–1 E***

An object with horizontal velocity 2 ft/sec begins to fall. How far will it have fallen when it has moved 5 feet horizontally?

$$v = 2 \text{ ft/sec}, x = 5: \quad y = -\frac{16}{v^2}x^2$$

$$y = -\frac{16}{2^2} \cdot 5^2 = -4(25) = -100$$

Thus, the object will have fallen 100 feet when it has moved 5 feet horizontally. ■

Mastery points

Can you

- Graph an equation of the form $y = ax^2 + bx + c$ or $x = y^2 + by + c$ as a parabola?
- Find the equation of a parabola, given certain initial conditions?

Exercise 11–1

Graph each parabola; clearly state the focus and directrix. State the intercepts and vertex where not clear from the graph.

1. $y = -2x^2$
2. $y = \frac{1}{2}x^2$
3. $y = 3x^2$
4. $y = -\frac{1}{3}x^2$
5. $y = x^2 - 4$
6. $y = 2x^2 - 4$
7. $y = -x^2 + 1$
8. $y = x^2 - 9$
9. $y = 2(x - 3)^2$
10. $y = 3(x + 1)^2$
11. $y = \frac{1}{2}(x + 2)^2$
12. $y = -\frac{1}{2}(x - 2)^2$
13. $y = -(x + 1)^2$
14. $y = (x + 2)^2 + 3$
15. $y = 3(x - \frac{1}{2})^2 - 1$
16. $y = 2(x - 1)^2 - 1$
17. $y = -2(x + 2)^2 + 1$
18. $y = x^2 - 4x - 3$
19. $y = x^2 + 5x + 15$
20. $y = -x^2 + 5x + 10$
21. $y = -x^2 + 6x - 7$
22. $y = x^2 - 6x + 9$
23. $y = x^2 + 2x + 1$
24. $y = 2x^2 + 5x - 3$
25. $y = 2x^2 - x - 3$
26. $y = -3x^2 - 10x + 8$
27. $y = -x^2 + 16$
28. $y = 9 - x^2$
29. $y = x^2 - 3x - 5$
30. $y = x^2 + 4x + 1$
31. $y = 2x^2 - 6x - 1$
32. $y = 3x^2 + 9x + 11$
33. $y = -3x^2 - 4x + 7$
34. $y = -x^2 + 5x + 10$

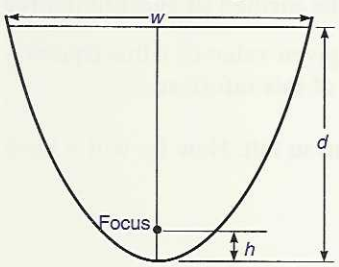
Graph each parabola; state the focus and directrix. State the intercepts and vertex where not clear from the graph.

35. $x = y^2 - 7y - 8$
36. $x = y^2 + 3y - 18$
37. $x = y^2 - 9$
38. $x = y^2 - 4y + 4$
39. $x = -y^2 + y + 2$
40. $x = -y^2 - 4y + 8$

Find the equation of a parabola with the given properties.

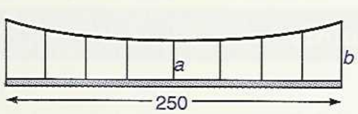
41. focus: $(2,-3)$; directrix: $y = -6$
42. focus: $(-1,4)$; directrix: $y = 0$
43. focus: $(-3,-1)$; directrix: $y = 2$
44. focus: $(0,2)$; directrix: $y = 5$
45. vertex: $(3,-1)$; directrix: $y = -3$
46. vertex: $(-2,\frac{1}{3})$; directrix: $y = 1$
47. focus: $(0,3)$; vertex: $(0,0)$
48. focus: $(-4,2)$; vertex: $(-4,4)$
49. vertex: $(3,-1)$; x-intercepts: 2, 4
50. vertex: $(-2,-3)$; y-intercept: -1; opens vertically

Refer to the figure for problems 51–56. Assume all values are in inches.



51. $h = 8$, $d = 24$; find w.
52. $h = 3$, $d = 9$; find w.
53. $h = 5$, $w = 12$; find d.
54. $h = 20$, $w = 50$; find d.
55. $w = 24$, $d = 20$; find h.
56. $w = 30$, $d = 34$; find h.

57. The cable on a suspension bridge takes the form of a parabola. The figure shows such a bridge, which is 250 feet long. At a it's height is 30 feet. At b it is 45 feet. Find an equation that would describe this parabola assuming the low point of the cable is at the origin.

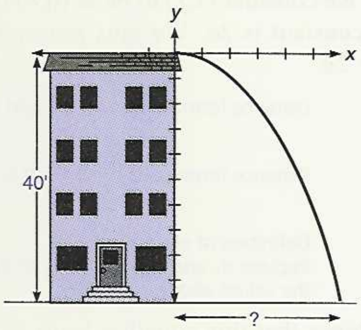


58. The path of an object launched near the surface of the earth at a velocity of $8\sqrt{2}$ feet per second and at an angle of 30° would follow the path described by $y = \frac{1}{\sqrt{3}}x - \frac{1}{6}x^2$, $x \geq 0$. Graph this parabola for those values for which $x \geq 0$ and $y \geq 0$. Label the intercepts and vertex.

59. As in problem 58, if the angle at which the object were launched were 45°, the trajectory of the object would be described by $y = x - \frac{1}{4}x^2$, $x \geq 0$. Graph this parabola for those values for which $x \geq 0$ and $y \geq 0$. Label the intercepts and vertex.

Use the formula $y = -\frac{16}{v^2}x^2$ for problems 60 through 64.

$\boxed{60.}$ A stunt person is going to jump off a 40-foot-high building, running at an estimated horizontal velocity v of 4 feet per second. How far from the base of the building will this person land?



61. If the same individual (problem 60) runs at 8 feet per second how far from the building will this person land? The horizontal velocity doubled (4 ft/s to 8 ft/s); did the horizontal distance traveled double also?

62. What would be the required velocity to land 10 feet from the base of the building of problem 60?

63. What would be the required velocity to land 4 feet from the base of the building of problem 60?

$\boxed{64.}$ For a fixed horizontal velocity, does doubling the height of the fall also double the horizontal distance of the landing point from the base of the building?

Skill and review

1. Solve $\frac{x^2}{4} + 3y^2 = 1$ for y.

2. Multiply the matrices $\begin{bmatrix} -2 & 3 & 0 \\ 1 & 5 & -3 \\ 4 & 2 & 6 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & 5 \\ -3 & 2 \end{bmatrix}$.

3. Solve the system
$2x + 3y - z = 5$
$4x - 6y + z = -4$
$2x + 6y - 3z = 11$

4. Solve the system $\begin{matrix} 2x - y \le 4 \\ x + y \ge 3 \end{matrix}$.

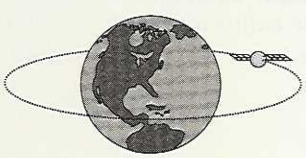
5. Compute $\log_5 10$ to 4 decimal places.

6. Solve $\log(2x - 1) + \log(3x + 1) = \log 4$.

7. Solve $|2x - 5| < 10$.

11–2 The ellipse

An asteroid has been found orbiting the sun with an elliptical orbit such that the sun is at one focus, and the asteroid is 200 million miles from the sun at its farthest point, and 100 million miles from the sun at its closest point. Find an equation that describes the path of the asteroid.



An ellipse is a ''flattened circle.'' It is the path that the earth takes around the sun and that satellites take around the earth. As stated in this problem it is the path that an asteroid takes when it goes around the sun. This fact was first discovered by the astronomer Johann Kepler (1571–1630).

Like the parabola the ellipse can be defined from a geometrical viewpoint. Fix two points, called the foci (c and $-c$ in figure 11–5). Now find all points such that the sum of the two distances from that point to each focus ($d_1 + d_2$ in the figure) is a constant.

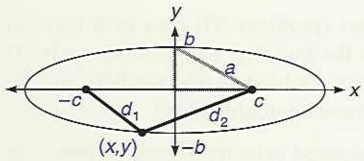


Figure 11–5

Figure 11–5 shows an ellipse placed so its center is at the origin. The foci are placed on the x-axis, equidistant from the origin; they are at $(-c,0)$ and $(c,0)$. The point (x,y) represents any point on the ellipse. The y-intercept is labeled b. We call the distance from the y-intercept to the focus a. The right triangle shown illustrates that $a^2 = b^2 + c^2$. We can develop an analytic description of this ellipse as follows.

The sum of d_1 and d_2 is a constant. If we consider (x,y) to be at $(0,b)$ (one of the y-intercepts) we can see that this constant is $2a$. We thus proceed algebraically from the statement $d_1 + d_2 = 2a$.

$d_1 = \sqrt{(x-(-c))^2 + (y-0)^2}$	Distance formula with $(-c,0)$ and (x,y)
$= \sqrt{(x+c)^2 + y^2}$	
$d_2 = \sqrt{(x-c)^2 + (y-0)^2}$	Distance formula with $(c,0)$ and (x,y)
$= \sqrt{(x-c)^2 + y^2}$	
$d_1 + d_2 = 2a$	Definition of ellipse
$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$	Replace d_1 and d_2 in $d_1 + d_2 = 2a$ by the values above

Appendix A describes the algebra that shows that this equation leads to the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where we define b so that $b^2 = c^2 - a^2$.

By letting x and then y be zero we find that the y-intercept is at b (which we knew already), and the x-intercept is at a. Also, solving for c in $a^2 + b^2 = c^2$, we find that $c = \sqrt{a^2 - b^2}$. The **foci** in this development are at $x = \pm c$. By our definition of a, we guaranteed that $a > b$. *It can be shown that if $a < b$ the foci are on the y-axis and $c = \sqrt{b^2 - a^2}$.* We thus obtain the following result.

> **Ellipse**
> The graph of an equation of the form
> $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
> is an ellipse whose center is at the origin.
> - If $a > b$, then $c = \sqrt{a^2 - b^2}$ and the foci are at $(\pm c,0)$.
> - If $a < b$, then $c = \sqrt{b^2 - a^2}$ and the foci are at $(0,\pm c)$.

A **line segment** is a part of a line with a finite length. The line segment along the axis on which the foci lie is called the **major axis;** the other axis is the **minor axis** (see figure 11–6). *The length of the major axis is the greater of 2a and 2b, and the length of the minor axis is the lesser of these two values.*

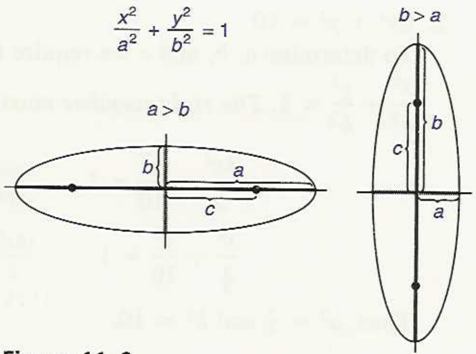


Figure 11–6

> **To graph an ellipse in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$**
>
> - The major axis is along the x-axis if $a > b$ and along the y-axis if $b > a$.
> - Plot the end points of the major and minor axes. These are also the x- and y-intercepts.
> - $c = \sqrt{a^2 - b^2}$ if $a > b$ or $c = \sqrt{b^2 - a^2}$ if $b > a$.
> - The foci are c units from the center along the major axis.

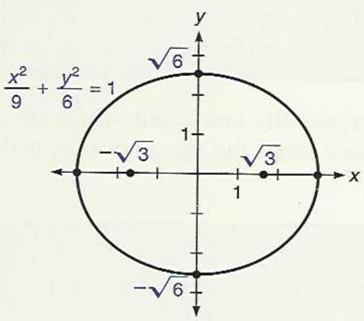
It is not possible to graph an ellipse whose equation is given in the form here by simply entering it into a graphing calculator. A method for using the graphing calculator is discussed at the end of this section.

Example 11–2 A illustrates graphing ellipses with center at the origin.

■ *Example 11–2 A*

Graph each ellipse.

1. $\frac{x^2}{9} + \frac{y^2}{6} = 1$



y-intercepts: Let $x = 0$.

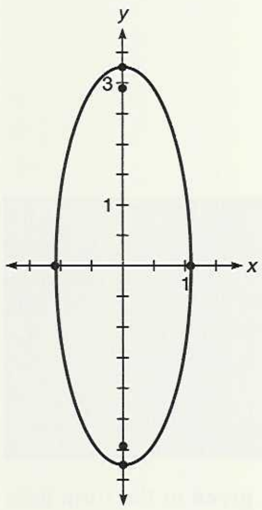
$$\frac{0^2}{9} + \frac{y^2}{6} = 1$$
$$\frac{y^2}{6} = 1$$
$$y^2 = 6$$
$$y = \pm\sqrt{6} \approx \pm 2.4$$

x-intercepts: Let $y = 0$.

$$\frac{x^2}{9} + \frac{0^2}{6} = 1$$
$$\frac{x^2}{9} = 1$$
$$x^2 = 9$$
$$x = \pm 3$$

$$c = \sqrt{a^2 - b^2} = \sqrt{9 - 6} = \sqrt{3}$$

Since the x-intercepts are farther apart than the y-intercepts, the x-axis contains the major axis. This is therefore where the foci are.



2. $8x^2 + y^2 = 10$

To determine *a, b,* and *c* we require that the equation be in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. *The right member must be 1.*

$$\frac{4x^2}{5} + \frac{y^2}{10} = 1$$ Divide each term by 10 so the right member becomes 1

$$\frac{x^2}{\frac{5}{4}} + \frac{y^2}{10} = 1$$ $\dfrac{(4x^2) \div 4}{5 \div 4} = \dfrac{x^2}{\frac{5}{4}}$

Thus, $a^2 = \frac{5}{4}$ and $b^2 = 10$.

y-intercepts:

$$\frac{0^2}{\frac{5}{4}} + \frac{y^2}{10} = 1$$

$$y^2 = 10$$

$$y = \pm\sqrt{10} \approx \pm 3.2$$

x-intercepts:

$$\frac{x^2}{\frac{5}{4}} + \frac{0^2}{10} = 1$$

$$x^2 = \frac{5}{4}$$

$$x = \pm\sqrt{\frac{5}{4}} = \pm\frac{\sqrt{5}}{2} \approx \pm 1.1$$

$$c = \sqrt{b^2 - a^2}$$

$$c = \sqrt{10 - \frac{5}{4}} = \sqrt{\frac{40}{4} - \frac{5}{4}} = \sqrt{\frac{35}{4}} = \frac{\sqrt{35}}{2} \approx 2.96$$

The *y*-axis contains the major axis since $b > a$. ■

A more general form of the equation of an ellipse takes horizontal and vertical translations into account.

> **General form of the equation of an ellipse**
> $$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

This is an ellipse that is translated *h* units horizontally and *k* units vertically. Thus its *center* is at (h,k) instead of $(0,0)$. In this form, the values of *a, b,* and *c* are distances from the center (h,k).

> **To graph any ellipse**
> - The major axis is parallel to the *x*-axis if $a > b$ and parallel to the *y*-axis if $b > a$.
> - Plot the end points of the major axis and minor axis *a* and *b* units, as appropriate, from (h,k).
> - $c = \sqrt{a^2 - b^2}$ if $a > b$ or $c = \sqrt{b^2 - a^2}$ if $b > a$.
> - The foci are *c* units from the center (h,k) along the major axis.

This is illustrated in example 11–2 B.

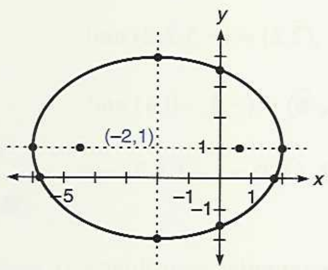
■ *Example 11–2 B*

Graph the ellipse $\frac{(x+2)^2}{16} + \frac{(y-1)^2}{9} = 1$. Label the points at the end of the major and minor axes, center, and foci.

$$\frac{(x-(-2))^2}{16} + \frac{(y-1)^2}{9} = 1 \qquad \text{Rewrite in the general form}$$

Center: $(-2,1)$ $\qquad (h,k) = (-2,1)$

Since $16 > 9$ the major axis is parallel to the x-axis.



End points of the major axis: $a = \sqrt{16} = 4$, so the end points of the major axis are 4 units to the right and left of the center $(-2,1)$. These are at $(-2 \pm 4,1)$ or $(2,1)$ and $(-6,1)$.

End points of the minor axis: $b = \sqrt{9} = 3$, so there are points 3 units above and below the center. These are $(-2,1 \pm 3)$ or $(-2,4)$ and $(-2,-2)$.

Foci: $c = \sqrt{16-9} = \sqrt{7}$, and the major axis is parallel to the x-axis, so the foci are $\pm\sqrt{7}$ right and left of the center. These are $(-2 \pm \sqrt{7},1)$ or $(-2-\sqrt{7},1)$ and $(-2+\sqrt{7},1)$, or about $(-4.6,1)$ and $(0.6,1)$. ■

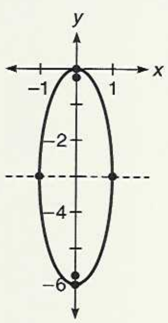
By completing the square we can put any equation of the form

$$Ax^2 + Bx + Cy^2 + Dy + F = 0$$

into the general form of an ellipse *if A and B both have the same sign.*[4] If the right side is positive we then have the equation of an ellipse. Example 11–2 C illustrates this procedure.

■ *Example 11–2 C*

Convert each equation into the standard form of the equation of an ellipse. Identify the points that terminate the major and minor axes and the foci, then graph.



1. $9x^2 + y^2 + 6y = 0$

$$9x^2 + y^2 + 6y + 9 = 9 \qquad \text{Complete the square on } y$$
$$9x^2 + (y+3)^2 = 9$$
$$x^2 + \frac{(y+3)^2}{9} = 1 \qquad \text{Divide each term by 9}$$

Center: $(0,-3)$, $a = 1$, $b = \sqrt{9} = 3$

The major axis is parallel to the y-axis since $b > a$.

End points of major axis: $(0,-3 \pm 3)$ or $(0,-6)$ and $(0,0)$

End points of minor axis: $(0 \pm 1,-3)$ or $(-1,-3)$ and $(1,-3)$

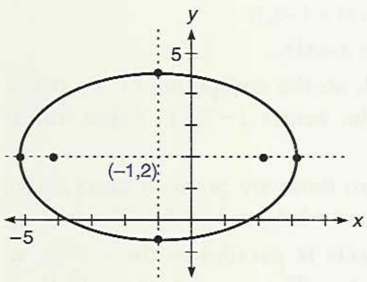
Foci: (Along the major (vertical) axis)

$c = \sqrt{9-1} = \sqrt{8} = 2\sqrt{2}$

Foci are $(0,-3 \pm 2\sqrt{2}) \approx (0,-5.8)$ and $(0,-0.2)$.

[4]If A and B have opposite signs, the figure is a hyperbola, covered in section 11–3.

2. $x^2 + 2x + 3y^2 - 12y - 5 = 0$
$x^2 + 2x + 3(y^2 - 4y) = 5$
$x^2 + 2x + 1 + 3(y^2 - 4y + 4) = 5 + 1 + 3(4)$
$(x + 1)^2 + 3(y - 2)^2 = 18$
$$\frac{(x + 1)^2}{18} + \frac{(y - 2)^2}{6} = 1$$



Center: $(-1,2)$, $a = \sqrt{18} = 3\sqrt{2}$, $b = \sqrt{6}$
Major axis: parallel to x-axis because $a > b$.
End points of major axis: $a = 3\sqrt{2}$; $(-1 \pm 3\sqrt{2},2) \approx (-5.2,2)$ and $(3.2,2)$
End points of minor axis: $b = \sqrt{6}$; $(-1,2 \pm \sqrt{6}) \approx (-1,-0.4)$ and $(-1,4.4)$
Foci: $c = \sqrt{18 - 6} = \sqrt{12} = 2\sqrt{3}$; $(-1 \pm 2\sqrt{3},2) \approx (-4.5,2)$ and $(2.5,2)$. ■

To graph any equation that is in terms of rectangular coordinates (x and y) on a graphing calculator, the relation (equation) must be solved for y. This is how we can use a graphing calculator to graph ellipses. Example 11–2 D illustrates.

■ Example 11–2 D

Graph the ellipse on a graphing calculator.

$\frac{(x + 2)^2}{16} + \frac{(y - 1)^2}{9} = 1$ (example 11–2 B)

$$\frac{(y - 1)^2}{9} = 1 - \frac{(x + 2)^2}{16}$$
$$(y - 1)^2 = 9\left(1 - \frac{(x + 2)^2}{16}\right)$$
$$y - 1 = \pm 3\sqrt{1 - \frac{(x + 2)^2}{16}}$$
$$y = 1 \pm 3\sqrt{1 - \frac{(x + 2)^2}{16}}$$

We will store the expression $3\sqrt{1 - \frac{(x + 2)^2}{16}}$ in Y_1, then let $Y_2 = 1 + Y_1$, and $Y_3 = 1 - Y_1$. We also must turn off Y_1, so it will not be graphed.

$\boxed{\text{Y=}}$ 3 $\boxed{\sqrt{\ }}$ $\boxed{(}$ 1 $\boxed{-}$ $\boxed{(}$ $\boxed{\text{X|T}}$ $\boxed{+}$ 2 $\boxed{)}$ $\boxed{x^2}$ $\boxed{\div}$ 16 $\boxed{)}$ $\boxed{\text{ENTER}}$

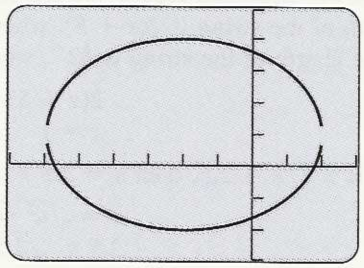
1 $\boxed{+}$ $\boxed{\text{Y-VARS}}$ $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$

1 $\boxed{-}$ $\boxed{\text{Y-VARS}}$ $\boxed{\text{ENTER}}$

Now use the up and down arrow keys to position the cursor on the "=" in "Y_1=," then select $\boxed{\text{ENTER}}$. This turns off Y_1 so it will not be graphed.

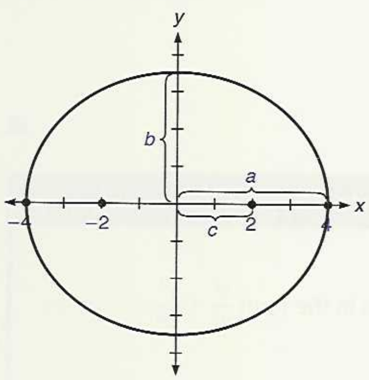
$\boxed{\text{RANGE } -7,3,-3,5}$

The graph appears to have holes at each end. This is due to the inherent inaccuracies of the calculator.



There are times when we wish to find the equation of an ellipse, given information such as the foci or intercepts, as illustrated in example 11–2 E.

■ Example 11–2 E



In a certain ellipse the x-intercepts are at ± 4 and the foci are on the x-axis at ± 2. Find the equation of the ellipse.

The major axis is the x-axis since the foci are on this axis. Therefore, $a = 4$ and $c = 2$.

$$\begin{aligned} c &= \sqrt{a^2 - b^2} \\ 2 &= \sqrt{4^2 - b^2} \\ 4 &= 16 - b^2 \\ b^2 &= 12 \end{aligned}$$

We put these values into the general equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to obtain $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

One way to actually draw an ellipse is to put two tacks in a drawing board, put a loop of string around the tacks, and draw the figure with the string stretched taught (figure 11–7). The tacks are at the foci, and the constant sum depends on the length of the string and the distance between the foci (tacks). Example 11–2 F illustrates this idea.

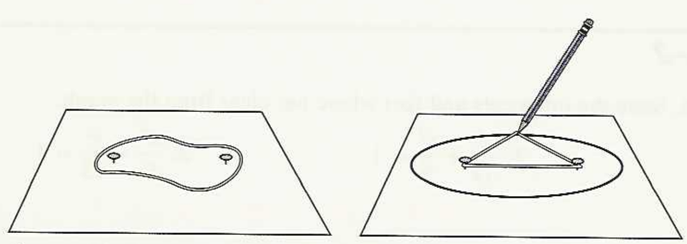
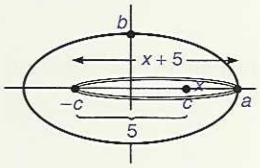


Figure 11–7

■ Example 11–2 F



Two tacks are put in a board (as in figure 11–7) 5 inches apart and a loop of string with length 12 inches is looped around the tacks. The ellipse is drawn. Find an equation that describes the ellipse, assuming the center of the ellipse is at the origin.

We can find the value of a, the x-intercept, as follows. Consider the loop of string shown, stretched around the left focus and the point a. We can see that the length of the string is $2(x + 5)$, where x is the distance between a and c. Since the length of the string is 12″, we solve for x:

$$2(x + 5) = 12$$
$$x = 1$$

Since c is $5 \div 2 = 2.5$, then $a = 2.5 + 1 = 3.5$.

$$c = \sqrt{a^2 - b^2}$$
$$2.5 = \sqrt{3.5^2 - b^2}$$
$$b^2 = 6$$

Plugging the values of $a^2 = 12.25$ and $b^2 = 6$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we obtain

$$\frac{x^2}{12\frac{1}{4}} + \frac{y^2}{6} = 1$$
$$\frac{x^2}{\frac{49}{4}} + \frac{y^2}{6} = 1$$
$$\frac{4x^2}{49} + \frac{y^2}{6} = 1$$

■

Mastery points

Can you

- Graph an ellipse when given its equation in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or the general form $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$?
- Convert certain equations, that are quadratic in both variables x and y into the general form $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$?
- Find the equation of an ellipse given certain conditions?

Exercise 11–2

Graph the equation. State the intercepts and foci where not clear from the graph.

1. $\frac{x^2}{16} + \frac{y^2}{9} = 1$ **2.** $\frac{x^2}{12} + \frac{y^2}{9} = 1$ **3.** $\frac{x^2}{9} + \frac{y^2}{25} = 1$ **4.** $\frac{x^2}{4} + \frac{y^2}{5} = 1$

5. $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **6.** $\frac{x^2}{16} + \frac{y^2}{25} = 1$ $\boxed{7.}$ $\frac{4x^2}{5} + y^2 = 1$ **8.** $x^2 + \frac{25y^2}{4} = 1$

9. $\frac{x^2}{16} + y^2 = 1$ **10.** $x^2 + \frac{y^2}{9} = 1$ **11.** $\frac{x^2}{49} + \frac{y^2}{25} = 1$ **12.** $\frac{x^2}{121} + \frac{y^2}{36} = 1$

Graph the equation. State the points at the end of the major and minor axes, center, and foci where not clear from the graph.

13. $\frac{(x+3)^2}{4} + \frac{(y-1)^2}{16} = 1$ **14.** $\frac{(x+1)^2}{4} + \frac{(y-2)^2}{9} = 1$ $\boxed{15.}$ $\frac{(x-2)^2}{25} + \frac{(y+3)^2}{9} = 1$

16. $\frac{(x-2)^2}{4} + (y+1)^2 = 1$ **17.** $\frac{(x-1)^2}{36} + (y+2)^2 = 1$ **18.** $(x-3)^2 + \frac{(y+1)^2}{36} = 1$

19. $\frac{(x+3)^2}{9} + \frac{y^2}{25} = 1$ **20.** $\frac{(x-2)^2}{100} + \frac{(y-3)^2}{49} = 1$

Convert each equation into standard form if necessary. Then graph the equation, stating the points at the end of the major and minor axes, center, and foci where not clear from the graph.

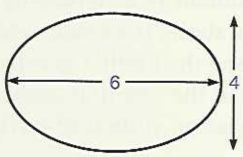
21. $x^2 + 3y^2 = 27$ **22.** $3x^2 + 4y^2 = 12$ **23.** $36x^2 + 9y^2 = 324$ **24.** $16x^2 + y^2 = 64$
25. $9x^2 + 2y^2 = 18$ **26.** $8x^2 + y^2 = 16$ **27.** $6x^2 + 3y^2 = 27$ **28.** $4x^2 + y^2 = 4$
29. $x^2 + 9y^2 = 9$ **30.** $9x^2 + 4y^2 = 36$ **31.** $12x^2 + 6y^2 = 12$ **32.** $5x^2 + 3y^2 = 15$
33. $4x^2 + 5y^2 = 5$ **34.** $3x^2 + y^2 = 3$ **35.** $x^2 + 2y^2 - 8y = 0$ **36.** $2x^2 - 4x + y^2 = 9$
$\boxed{37.}$ $4x^2 - 4x + 8y^2 + 48y = -57$ **38.** $2x^2 - 8x + y^2 = 0$ **39.** $x^2 - 6x + 2y^2 + 20y = 1$
40. $x^2 + 2x + 2y^2 + 12y + 1 = 0$ **41.** $x^2 + 2y^2 + 8 = 0$ **42.** $9x^2 + 4y^2 = 0$
43. $4x^2 - 8x + 9y^2 + 36y + 4 = 0$ **44.** $x^2 + 4x + 4y^2 - 8y + 4 = 0$ **45.** $16x^2 + 25y^2 + 100y = 300$
46. $2x^2 + 4x + 8y^2 = 30$

Find the equation of the ellipse with the required properties in problems 47–52.

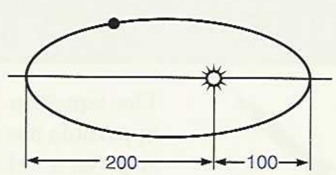
$\boxed{47.}$ foci: $(-2,0)$ and $(2,0)$; one y-intercept at 3
48. foci: $(-3,0)$ and $(3,0)$; one x-intercept at 5
49. foci: $(0,4)$ and $(0,-4)$; one y-intercept at 8
50. foci: $(0,1)$ and $(0,-1)$; one x-intercept at 1
$\boxed{51.}$ x-intercepts: $(\pm3,0)$; y-intercepts: $(\pm2,0)$
52. x-intercepts: $(\pm2,0)$; y-intercepts: $(\pm4,0)$

53. Two tacks are put in a board 4 inches apart and a string tied in a loop with length 10 inches is looped around the tacks. The ellipse is drawn. Find an equation that describes the ellipse.

$\boxed{54.}$ It is desired to construct an ellipse with height (along the minor axis) of 4 feet and length (along the major axis) of 6 feet. (See the figure.) Find out where to place the two tacks for the foci and how long to make the length of string.

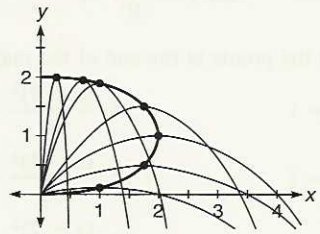


55. An asteroid has been found orbiting the sun with an elliptical orbit such that the sun is at one focus, and the asteroid is 200 million miles from the sun at its farthest point, and 100 million miles from the sun at its closest point. Find an equation that describes this orbit, using the values 200 and 100, as shown in the figure.



56. (Refer to problem 55.) Find an equation that describes the orbit of the asteroid if the distance from the sun was always the same, 200 million miles. Thus, the nearest and farthest points are both 200 in this case.

57. When water leaves a garden hose nozzle, its path through the air is a parabola (if we neglect air resistance; see the figure, where we picture the hose nozzle at the origin). As the direction of the nozzle is raised from the horizontal, the path traced by the highest point of the water's path is part of an ellipse.[5] If the water's velocity is $8\sqrt{2}$ feet per second the paths will be those shown in the figure. The ellipse shown has center at (0,1), minor axis of length 2 and major axis of length 4. Find it's equation.



[5]From a problem proposed by Alan Wayne in the May, 1989 issue of *The College Mathematics Journal.*

The *eccentricity* e of an ellipse is defined as the ratio $\frac{c}{a}$ when $a > b$, or $\frac{c}{b}$ when $b > a$. Examining the case where $a > b$, $a^2 = b^2 + c^2$, we know that $c < a$. As $0 \le c < a$, the ratio $e = \frac{c}{a}$ varies from 0 to 1: $0 \le e < 1$. When $e = 0$, the ellipse is a circle; as e approaches 1 the ellipse gets flatter and flatter. Find the eccentricity of the ellipse in each problem.

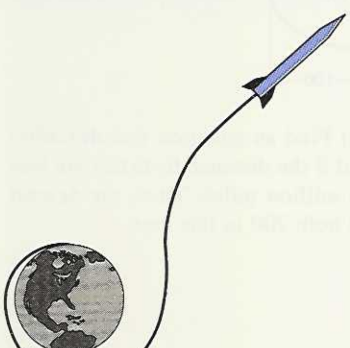
58. Problem 1.	**59.** Problem 2.	$\boxed{\textbf{60.}}$ Problem 3.	**61.** Problem 4.	$\boxed{\textbf{62.}}$ Problem 5.
63. Problem 6.	**64.** Problem 7.	**65.** Problem 8.	**66.** Problem 9.	**67.** Problem 10.
68. Problem 33.	**69.** Problem 34.	**70.** Problem 35.	**71.** Problem 36.	

Skill and review

1. Graph $y = 2(x - 1)^2 - 4$.
2. Graph $x^2 - 4x + y^2 + 12y + 12 = 0$.
3. Solve $x^{2/3} + 7x^{1/3} = 8$.
4. Rationalize the denominator: $\frac{2\sqrt{3}}{\sqrt{6} - \sqrt{2}}$.
5. Graph $f(x) = \frac{2x}{x^2 - 9}$.
6. Factor $x^3 - 3x^2 + x + 2$ over R.

11–3 The hyperbola

Two long-range navigation (LORAN) radio navigation stations are 130 miles apart. A ship receiving the signals from these stations determines that the difference in the distances from the ship to each station is 50 miles. Find the equation that describes this situation.



The equation that describes the situation in this problem is a hyperbola. A hyperbola has two parts, each of which resembles a parabola. If a space vehicle is given a velocity in excess of what it needs to escape the earth's gravity, it leaves the earth on a hyperbolic path. As suggested in the opening problem, hyperbolas also form the basis for the marine and aviation system of navigation called LORAN.

To define a hyperbola geometrically, fix two points, called the foci. (At c and $-c$ in figure 11–8.) For every point (x,y) there is a distance d_1 to one focus and a distance d_2 to the other. The **hyperbola** is the set of all points such that the absolute value of the difference between d_1 and d_2 is a constant.

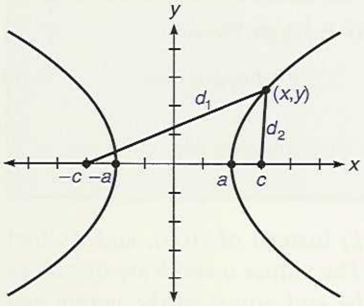


Figure 11–8

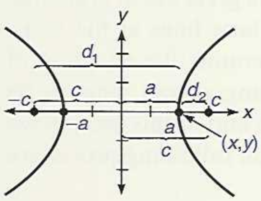


Figure 11–9

To develop an algebraic description of a hyperbola we proceed as follows. Consider figure 11–8. In this case, the foci are placed on the x-axis, at c and $-c$ $(c > 0)$. The x-intercepts are at a and $-a$, and (x,y) represents any point on the hyperbola. We are told that $|d_1 - d_2|$ is some constant. By letting the point (x,y) be at the point $(a,0)$ it can be seen that this constant is $2a$ (see figure 11–9).

$$|d_1 - d_2| = |(a + c) - (c - a)| = |2a|$$

If $a > 0$,

$$|d_1 - d_2| = 2a$$

Now find d_1 and d_2 with the distance formula.

$d_1 = \sqrt{(x - (- c))^2 + (y - 0)^2}$ Use the distance formula on (x,y) and $(-c,0)$

$d_2 = \sqrt{(x - c)^2 + (y - 0)^2}$ Use the distance formula on (x,y) and $(c,0)$

$|d_1 - d_2| = 2a$ Established above

$\left|\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2}\right| = 2a$ Substitute for d_1 and d_2

By carrying out the details of this calculation, and defining a value b so that $b^2 = c^2 - a^2$, the last statement can be transformed into $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. This transformation is left as an exercise. Observe that if $b^2 = c^2 - a^2$, then $c^2 = a^2 + b^2$; we use this below.

By letting x and then y be zero, we find that there is no y-intercept and the x-intercept is at a. The origin, with respect to any translated axes, is called the **center** of the hyperbola. By putting the foci on the y-axis we obtain a similar equation. We thus obtain the following result.

> **Hyperbola**
>
> The graph of an equation of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola with center at the origin. The hyperbola opens right and left. The x-intercepts are at $(\pm a,0)$, and there are no y-intercepts. The foci are at $(\pm c,0)$, where $c^2 = a^2 + b^2$.
>
> The graph of an equation of the form $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is a hyperbola with center at the origin. The hyperbola opens up and down. The y-intercepts are at $(0,\pm a)$, and there are no x-intercepts. The foci are at $(0,\pm c)$, where $c^2 = a^2 + b^2$.

A **major axis** and a **minor axis** is also defined. The major axis is the line segment between the foci. The minor axis is perpendicular to the major axis. It is the line segment extending $|b|$ units above and below the center.

As in the case of the ellipse, a more general form of the equation of a hyperbola takes horizontal and vertical translations into account.

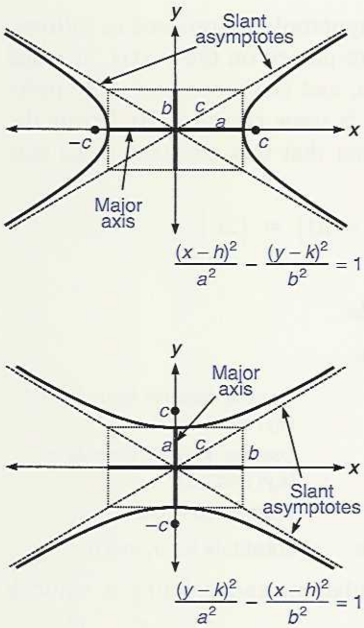


Figure 11–10

> **Standard form of the equation of a hyperbola**
>
> [1] $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ Opens horizontally
>
> [2] $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ Opens vertically

Each of these equations has its *center* at (h,k) instead of $(0,0)$, and its foci must be computed relative to this new center. The values a and b are *distances* from the center (h,k) at which *we can plot the end points of the major and minor axes.*

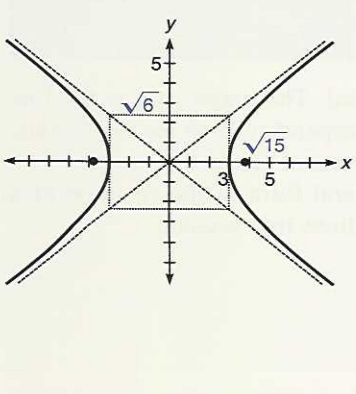
Hyperbolas also have **slant asymptotes,** which are a great aid in graphing. The graph of the hyperbola gets closer and closer to these lines as the value of $|x|$ gets larger and larger. We do not need to determine the equation of these asymptotes—the following procedure for graphing allows them to be constructed without determining their equations. At the end of this section we discuss why these lines are in fact slant asymptotes. The following procedure for graphing a hyperbola refers to figure 11–10.

> **To graph a hyperbola**
> - Find the values of a and b and the center (h,k). Determine whether the major axis is horizontal or vertical.
> - Draw a rectangle with sides a units from the center along the major axis and b units from the center along the minor axis.
> - Construct slant asymptotes. These are the lines that form the diagonals of the rectangle.
> - Compute c using $c = \sqrt{a^2 + b^2}$. The foci are c units from the center, along the major axis. Observe in the figure that c can also be viewed as the hypotenuse of a triangle with sides of length a and b.
> - Sketch the hyperbola using the rectangle and slant asymptotes as a guide.

Example 11–3 A illustrates graphing a hyperbola.

■ Example 11–3 A

Graph each hyperbola. State the foci and center. Show the rectangle that is used to draw the slant asymptotes.



1. $\frac{x^2}{9} - \frac{y^2}{6} = 1$

The center is at $(0,0)$. The major axis is horizontal, and the minor axis is vertical.

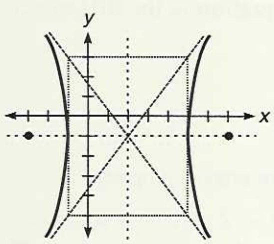
$$a = \sqrt{9} = 3$$
$$b = \sqrt{6} \approx 2.4$$

x-intercepts: $(\pm 3,0)$

Construct a rectangle as shown and draw the slant asymptotes.

$$c = \sqrt{a^2 + b^2} = \sqrt{9 + 6} = \sqrt{15} \approx 3.9$$

Since the major axis is horizontal the foci are along the x-axis at $(-\sqrt{15},0)$ and $(\sqrt{15},0)$. Draw the hyperbola so it touches the end points of the major axis and approaches the slant asymptotes.



2. $\dfrac{(x-2)^2}{9} - \dfrac{(y+1)^2}{16} = 1$

Center is at $(2,-1)$. The major axis is horizontal.

$$a = \sqrt{9} = 3; \qquad b = \sqrt{16} = 4$$

Construct a rectangle with sides 3 units right and left of the center and 4 units above and below it. Draw in the slant asymptotes.

$$c = \sqrt{a^2+b^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

The foci are 5 units right and left of $(2,-1)$ at $(2 \pm 5,-1)$, or $(-3,-1)$ and $(7,-1)$.

Note By completing the square we can put any equation of the form $Ax^2 + Bx + Cy^2 + Dy + E = 0$ into the standard form of a hyperbola if A and B have opposite signs.

3. $9y^2 + 18y - 4x^2 + 24x = 36$

$9(y^2 + 2y) - 4(x^2 - 6x) = 36$ — Factor 9 from y terms, -4 from x terms

$9(y^2 + 2y + 1) - 4(x^2 - 6x + 9) = 9(1) - 4(9) + 36$ — Complete the square

$9(y+1)^2 - 4(x-3)^2 = 9$

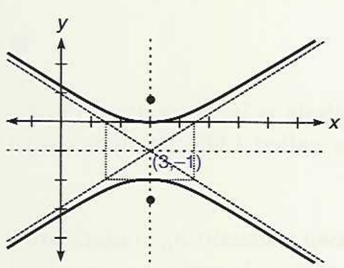
$\dfrac{9(y+1)^2}{9} - \dfrac{4(x-3)^2}{9} = \dfrac{9}{9}$ — Divide each term by 9 to obtain 1 in the right member

$(y+1)^2 - \dfrac{(x-3)^2}{\frac{9}{4}} = 1$

Center is at $(3,-1)$; major axis is vertical.

$$a = \sqrt{1} = 1; \qquad b = \sqrt{\tfrac{9}{4}} = \tfrac{3}{2} = 1.5$$

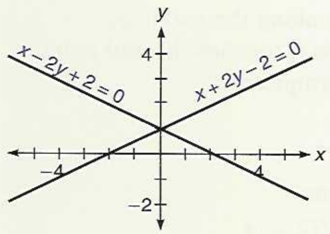
End points of major axis: $(3,-1 \pm 1)$ or $(3,-2)$ and $(3,0)$
End points of minor axis: $(3 \pm 1.5,-1)$ or $(1.5,-1)$ and $(4.5,-1)$



$$c = \sqrt{a^2+b^2} = \sqrt{1+\tfrac{9}{4}} = \sqrt{\tfrac{13}{4}} = \frac{\sqrt{13}}{2} \approx 1.8$$

Foci: $\left(3,-1 \pm \dfrac{\sqrt{13}}{2}\right) \approx (3,-2.8)$ and $(3,0.8)$

Draw a rectangle using these end points, and draw the slant asymptotes. Then draw the hyperbola so it touches the end points of the major axis and approaches the slant asymptotes.



4. $x^2 - 4y^2 + 8y = 4$

$x^2 - 4(y - 1)^2 = 0$ Coefficient of y^2 is 1

This equation cannot be put into the standard form of a hyperbola since the right member will never be 1. However, the equation is the difference of two squares and will factor.

$$x^2 - 4(y-1)^2 = 0$$
$$[x - 2(y-1)][x + 2(y-1)] = 0 \qquad m^2 - n^2 = (m-n)(m+n)$$
$$(x - 2y + 2)(x + 2y - 2) = 0$$
$$x - 2y + 2 = 0 \text{ or } x + 2y - 2 = 0 \qquad \text{Zero product property}$$

Each of the equations $x - 2y + 2 = 0$ and $x + 2y - 2 = 0$ is a straight line. This could be considered a "degenerate" hyperbola. ■

As with the ellipse, we must solve an equation for y to graph it with a graphing calculator.

■ ***Example 11–3 B***

Graph with a graphing calculator.

$$(y+1)^2 - \frac{(x-3)^2}{\frac{9}{4}} = 1 \qquad \text{(part 3, example 11–3 A.)}$$

$$(y+1)^2 = \tfrac{4}{9}(x-3)^2 + 1$$
$$y + 1 = \pm\sqrt{\tfrac{4}{9}(x-3)^2 + 1}$$
$$y = -1 \pm \sqrt{\tfrac{4}{9}(x-3)^2 + 1}$$

Put the expression $\sqrt{\frac{4}{9}(x-3)^2 + 1}$ into Y_1, then let $Y_2 = -1 - Y_1$, and $Y_3 = -1 + Y_2$.

[Y=] [√] [(] [(] 4 [÷] 9 [)] [(] [X|T] [−] 3 [)] [x^2]
[+] 1 [)] [ENTER] [(−)] 1 [−] [Y-VARS] 1 [ENTER]
[(−)] 1 [+] [Y-VARS] 1

Now turn off the graphing of Y_1 by positioning the cursor on the "=" part of "Y_1=" and selecting [ENTER].

[RANGE −1,7,−4,2] ■

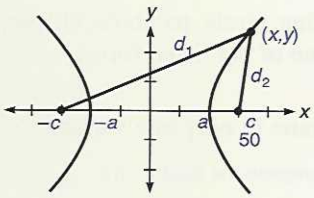
One of the most common uses of the hyperbola is in navigation. In particular, a system of navigation by radio waves called LORAN (long-range navigation) is based on the hyperbola.[6]

■ ***Example 11–3 C***

In the LORAN radio-aided navigation system, two transmitting stations are the foci of a hyperbola, and the receiving radio is on a hyperbola determined by the foci and the location of a ship or aircraft.

Two LORAN radio navigation stations are 100 miles apart. A ship receiving the signals from these stations determines that the difference in the

[6]Other hyperbolic navigation systems are the Decca Navigation System, Omega, and the satellite-based Global Positioning System.



distances from the ship to each station is 80 miles. Find the equation of a hyperbola that describes the position of the ship.

The figure shows the ship at some point (x,y). We know that $|d_1 - d_2| = 80$. The stations are at foci 50 units from the origin (since they are 100 miles apart). Thus $c = 50$.

When we developed the equation of a hyperbola at the beginning of this section we noted that the difference between the distances is $2a$. Thus,

$2a = 80$, so $a = 40$

$c^2 = a^2 + b^2$; $50^2 = 40^2 + b^2$; $30 = b$

$\dfrac{x^2}{1{,}600} - \dfrac{y^2}{900} = 1$ Replace values in $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ ■

The general quadratic equation in two variables

The general quadratic equation in two variables is $Ax^2 + By^2 + Cx + Dy + E = 0$, where A and B are not both zero. The graph of this equation depends largely upon the values of A and B. We categorize these equations in the following way.

General quadratic equation in two variables:

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

- If A or B (but not both) is zero, the equation is a parabola.
- If $A = B$ (and neither is zero), then the equation is a circle.
- If $|A| \neq |B|$ but A and B have the same sign, the equation is an ellipse.
- If A and B have opposite signs the equation is a hyperbola.

In each case it is possible that no graph exists or that is corresponds to another geometric object. For example, the equation $x^2 + y^2 = -4$ falls in the category of a circle, but does not have a graph. The equation $x^2 + y^2 = 0$ also falls in the category of a circle, but its graph is the single point $(0,0)$. The best way to determine the category of an equation is to put it in one of the following forms, usually by completing the square.

Straight line:	$ax + by + c = 0$, a and b not both 0
Parabola:	$y = \dfrac{1}{4p}(x - h)^2 + k$ or $x = \dfrac{1}{4p}(y - k)^2 + h$
Circle:	$(x - h)^2 + (y - k)^2 = r^2$
Ellipse:	$\dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$
Hyperbola:	$\dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1$ or $\dfrac{(y - k)^2}{a^2} - \dfrac{(x - h)^2}{b^2} = 1$

Example 11–3 D shows the process of categorizing equations by their geometric significance.

Example 11–3 D

Categorize the graph of each equation as a point, line, circle, parabola, ellipse, or hyperbola (or no graph). Put the equation in one of the listed forms.

1. $3x^2 + 2x - 5y = 8$

This is most likely a parabola since it is quadratic in only one variable, x.

$y = \frac{3}{5}(x + \frac{1}{3})^2 - \frac{5}{3}$ Complete the square on x

The graph is a parabola.

2. $2x^2 - 3y^2 + 9y = 5\frac{3}{4}$

This is most likely a hyperbola since the coefficients of x^2 and y^2 have opposite signs.

$\dfrac{(y - \frac{3}{2})^2}{\frac{1}{3}} - \dfrac{x^2}{\frac{1}{2}} = 1$ Complete the square on y

This is a hyperbola.

3. $2x^2 + 4x + 2y^2 + 6 = 0$

This is probably a circle because the coefficients of x^2 and y^2 have the same sign and value.

$(x + 1)^2 + y^2 = -2$ Complete the square

The left member is nonnegative and the right member is negative; thus the graph of this equation has no points. ■

Equations of slant asymptotes

As already illustrated, slant asymptotes are a great aid in graphing hyperbolas. To see why these asymptotes exist consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If we solve for y^2, we obtain $y^2 = \frac{b^2}{a^2}x^2 - b^2$. As $|x|$ gets greater (graphically, as we move right and left of the origin), the term $\frac{b^2}{a^2}x^2$ gets greater and greater, but the term $-b^2$ is fixed in size. As a percentage of the total value of $\frac{b^2}{a^2}x^2 - b^2$, b^2 tends to become less and less. To see this, let $a = b = 1$, giving $x^2 - 1$ and x^2. The table shows how the percentage of error diminishes as $|x|$ grows. The last column shows the percentage of error between $x^2 - 1$ and x^2.

x	$\lvert x \rvert$	x^2	$x^2 - 1$	$\lvert$error$\rvert$	$\frac{\lvert \text{error} \rvert}{x^2 - 1} \times 100\%$
±10	10	100	99	1	1.00%
±20	20	400	399	1	0.25%
±30	30	900	899	1	0.11%
±100	100	10,000	9,999	1	0.01%

Even a 1% error is usually not detectable in a graph. Thus, the graph of $x^2 - 1$ and x^2 are practically the same when $|x|$ gets greater and greater.

Generalizing, we would say that as $|x|$ gets greater and greater, the difference between $\frac{b^2}{a^2}x^2 - b^2$ and $\frac{b^2}{a^2}x^2$ gets smaller and smaller (as a percentage of $\frac{b^2}{a^2}x^2 - b^2$). Continuing the development above,

$$y^2 \approx \frac{b^2}{a^2}x^2 \qquad \text{Ignore } -b^2 \text{ when } |x| \text{ is large}$$

$$y \approx \pm\frac{b}{a}x$$

The lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are the slant asymptotes for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The graph of the hyperbola gets closer and closer to these lines as the value of $|x|$ gets greater and greater.

Similar reasoning will show that $y - k = \frac{b}{a}(x - h)$ and $y - k = -\frac{b}{a}(x - h)$ are the equations of the slant asymptotes for a hyperbola of the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$. See exercises 70 and 71 also.

Mastery points

Can you

- Graph a hyperbola when given the standard equation
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ or } \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1?$$
- Transform certain equations that are quadratic in two variables into the standard form of the equation of a hyperbola?
- Categorize equations of the form $Ax^2 + By^2 + Cx + Dy + E = 0$ as lines, parabolas, circles, ellipses, and hyperbolas?

Exercise 11–3

Graph each hyperbola. If necessary transform the equation first. State the coordinates of the end points of the major axis and foci where not clear from the graph.

1. $\frac{x^2}{25} - \frac{y^2}{16} = 1$
2. $\frac{y^2}{20} - \frac{x^2}{16} = 1$
3. $\frac{y^2}{25} - \frac{x^2}{16} = 1$
4. $\frac{x^2}{20} - \frac{y^2}{16} = 1$
5. $x^2 - \frac{y^2}{6} = 1$
6. $\frac{y^2}{12} - x^2 = 1$
7. $\frac{x^2}{16} - \frac{y^2}{4} = 1$
8. $\frac{x^2}{4} - \frac{y^2}{25} = 1$
9. $\frac{y^2}{9} - \frac{x^2}{4} = 1$
10. $\frac{y^2}{16} - \frac{x^2}{9} = 1$
11. $\frac{4x^2}{25} - y^2 = 1$
12. $\frac{16y^2}{9} - x^2 = 1$
13. $4y^2 - x^2 = 2$
14. $6x^2 - 18y^2 = 36$
15. $9x^2 - y^2 = 36$
16. $y^2 - 25x^2 = 25$
17. $16y^2 - x^2 = -16$
18. $2y^2 - 3x^2 = -18$
19. $2x^2 - 9y^2 = -36$
20. $6x^2 - 6y^2 = -1$
21. $25y^2 - 16x^2 = 400$
22. $16x^2 - 9y^2 = 144$
23. $8x^2 - 3y^2 = 4$
24. $5y^2 - 4x^2 = 10$

25. $\dfrac{(x-2)^2}{100} - \dfrac{(y+3)^2}{25} = 1$
26. $\dfrac{(y-1)^2}{20} - (x+1)^2 = 1$
27. $\dfrac{(y+2)^2}{4} - x^2 = 1$
28. $\dfrac{(x-3)^2}{9} - \dfrac{(y+1)^2}{4} = 1$
29. $\dfrac{(x+1)^2}{25} - \dfrac{(y-1)^2}{36} = 1$
30. $\dfrac{(y-3)^2}{9} - (x-2)^2 = 1$
31. $(x-3)^2 - \dfrac{(y+1)^2}{16} = 1$
32. $y^2 - \dfrac{(x-5)^2}{25} = 1$
33. $(y-2)^2 - \dfrac{(x-2)^2}{4} = 1$
34. $\dfrac{4(x-1)^2}{25} - \dfrac{y^2}{4} = 1$
35. $\dfrac{16(y-1)^2}{25} - \dfrac{x^2}{9} = 1$
36. $\dfrac{25(x-1)^2}{36} - \dfrac{9(y+1)^2}{4} = 1$
37. $2x^2 - 4x - y^2 - 4y - 10 = 0$
38. $y^2 + 6y - 12x^2 = 3$
39. $4x^2 + 8x - 3y^2 + 24y + 4 = 0$
40. $x^2 - 2y^2 + 4y = 10$
41. $x^2 - 4x + 3y^2 - 24y + 49 = 0$
42. $3x^2 + 18x - 2y^2 - 4y + 19 = 0$
43. $x^2 - 2x - 4y^2 - 8y = 7$
44. $3x^2 + 24x - y^2 + 2y + 50 = 0$
45. $2y^2 - 12y - 4x^2 = 6$
46. $3x^2 - 12x - 2y^2 - 6 = 0$
47. $x^2 - 3x - 3y^2 + 6y = 0$
48. $5y^2 - 20y - 2x^2 + 2x = 5$

Categorize the graph of each equation as a point, line, circle, parabola, ellipse, or hyperbola (or no graph). Put each equation in the standard form for whichever geometric figure it represents. Graph each figure; state centers and foci as appropriate.

49. $x^2 + 2x - y^2 + 8y = 16$
50. $y^2 - 6y - x + 4 = 0$
51. $4x^2 - 8x - y^2 - 4y = 4$
52. $2y - 4x = 1$
53. $9y^2 - 4x^2 + 36 = 0$
54. $x^2 - 4x + y^2 - 8y + 16 = 0$
55. $2x^2 + 12x + y^2 - 8y + 32 = 0$
56. $4x^2 - 24x - 25y^2 - 50y = 14$
57. $4x^2 - 12x + 4y^2 + 20y + 30 = 0$
58. $3x - 2y + 8 = 0$
59. $25x^2 + 16y^2 = 100$
60. $x^2 - 2x + 2y^2 + 12y + 11 = 0$
61. $y = x^2 + 2x + 4$
62. $3x^2 + 4y^2 - 16y + 4 = 0$
63. $4x^2 + 8x + 4y^2 - 4y = 27$
64. $x^2 = 7y + 4$
65. $4y = 4x^2 - 20x + 23$
66. $x^2 - 2x + y^2 + 6y + 10 = 0$

67. Two LORAN radio navigation stations are 130 miles apart. A ship receiving the signals from these stations determines that the difference in the distances from the ship to each station is 50 miles. Find the equation of a hyperbola that describes this situation.

68. Two LORAN radio navigation stations are 80 miles apart. A ship receiving the signals from these stations determines that the difference in the distances from the ship to each station is 60 miles. Find the equation of a hyperbola that describes this situation.

69. Suppose that the cost y of producing t items is $y = \sqrt{t}$, and the profit x on t items is $x = \dfrac{\sqrt{t-4}-1}{2}$. Then $y^2 = t$, and

$$2x = \sqrt{t-4} - 1$$
$$4x^2 + 4x + 1 = t - 4$$
$$4x^2 + 4x + 5 = t$$

Replacing t by y^2, $y^2 = 4x^2 + 4x + 5$. Transform this relation and graph it.

70. In the text we saw that the lines $y = \pm\dfrac{b}{a}x$ are the slant asymptotes for the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Use a similar development to determine the equations of the slant asymptotes of a hyperbola of the form $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$.

71. In the text we saw that $y - k = \pm\dfrac{b}{a}(x - h)$ are the equations of the slant asymptotes for a hyperbola of the form $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$. Determine the equations of the slant asymptotes for a hyperbola of the form $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$.

72. In the text we saw that the definition of a hyperbola with foci at $(a,0)$ and $(-a,0)$ led to the statement

$$\left|\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right| = 2a$$

and stated that by carrying out the details of this calculation, it can be shown that this statement is equivalent to $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ where $b^2 = c^2 - a^2$. Complete this calculation, but, to make the calculations easier, start from the equivalent statement $\sqrt{(x+c)^2 + y^2} = \sqrt{(x-c)^2 + y^2} \pm 2a$.

Skill and review

Graph each relation.

1. $x^2 + y^2 - 8y = 0$
2. $3x^2 + 4y^2 = 12$
3. $3x - 2y = 6$
4. $2x^2 - 4x + 4y^2 = 2$
5. $y = x^2 - 6x - 8$

11–4 Systems of nonlinear equations and inequalities

If a rock is dropped into a well and a splash is heard after 3 seconds, how deep is the well?

The solution to this problem involves two separate equations—one that describes the rock as it falls into the well and another that describes the sound of the splash as it comes out of the well. The answer to the problem is found by solving the system of these two equations, using methods studied in this section.

Systems of nonlinear equations

Chapter 10 discussed systems of linear equations. In this section we discuss systems of two equations in two variables that may be nonlinear. To solve such a system means to find all the ordered pairs that satisfy each equation in the system. Geometrically this corresponds to the points of intersection of the graphs of each equation.

By way of example, consider the following. Hyperbolas form the theoretical basis for a system of navigation called LORAN (illustrated in section 11–3). This system is based on radio waves and charts. The idea is that a ship or aircraft receives radio signals from two transmitters. The ship does not know the distance to either transmitter, but can determine very accurately when each signal arrives at the ship. This permits computation of the time difference in the arrival of each signal, and this time interval corresponds to the distance the radio waves travel in that time interval. This difference tells the ship's navigator that the ship lies on a certain hyperbola. By doing the same thing with some other radio transmitter, it is known that the ship lies on some other hyperbola. *By determining where these hyperbolas intersect the position of the ship or aircraft is discovered.* Thus, locating the ship is equivalent to determining the point(s) of intersection of two hyperbolas. In practice this is done graphically, on special navigation charts, or by computers.

We use the method of substitution for expression to find these points. This method was first shown in section 3–2 and is restated here.

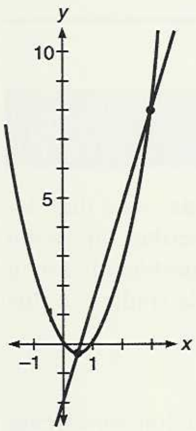
To solve a system of two equations using substitution for expression

- Solve one of the equations for one of the variables, say y. The other member of this equation is an expression in terms of x.
- In the *other* equation replace each instance of y by this expression in x.
- Solve this new equation for x.
- Use these values of x in either original equation to find y.

The role of x and y in these steps can be reversed when that is more convenient. Also we might solve for y^2 or some other expression in the first step. This method is illustrated in example 11–4 A.

Example 11–4 A

1. Find the point(s) where the line $3x - y = 2$ and the parabola $y = x^2 - x$ meet.



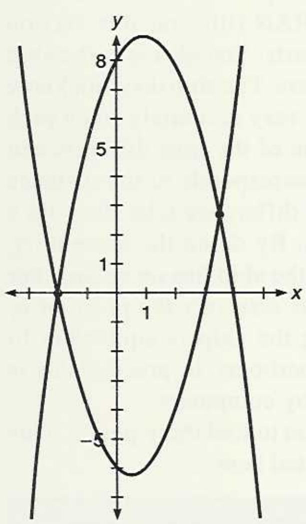
$3x - y = 2$	Equation of the line
$y = 3x - 2$	Solve for y
$y = x^2 - x$	Equation of parabola
$3x - 2 = x^2 - x$	Replace y with $3x - 2$ in $y = x^2 - x$
$0 = x^2 - 4x + 2$	Add $-3x + 2$ to each member
$x = 2 \pm \sqrt{2}$	Using the quadratic formula

This result (two values of x) implies that the line meets the parabola at two places. We still need the y values; these can be obtained from either the equation of the line, $y = 3x - 2$, or the parabola, $y = x^2 - x$. Either will give the same values of y. We use the line since it is easier.

x	$y = 3x - 2$
$2 + \sqrt{2}$	$y = 3(2 + \sqrt{2}) - 2 = 4 + 3\sqrt{2}$
$2 - \sqrt{2}$	$y = 3(2 - \sqrt{2}) - 2 = 4 - 3\sqrt{2}$

Thus, we find that there are two points where the line $y = 3x - 2$ meets the parabola $y = x^2 - x$; at $(2 + \sqrt{2}, 4 + 3\sqrt{2}) \approx (3.4,8.2)$, and at $(2 - \sqrt{2}, 4 - 3\sqrt{2}) \approx (0.6,-0.2)$. This is shown graphically in the figure.

2. Find the point(s) where the parabolas $y = x^2 - x - 6$ and $y = -x^2 + 2x + 8$ meet.



$y = x^2 - x - 6$	First equation; solved for y
$y = -x^2 + 2x + 8$	Second equation
$x^2 - x - 6 = -x^2 + 2x + 8$	Replace y in the second equation
$2x^2 - 3x - 14 = 0$	We want one of the members to be 0
$(2x - 7)(x + 2) = 0$	Factor
$2x - 7 = 0$ or $x + 2 = 0$	Zero product property
$x = 3\frac{1}{2}$ or -2	Solve each linear equation

Now compute the corresponding y values from either equation.

x	$y = x^2 - x - 6$	
$3\frac{1}{2}$	$y = (\frac{7}{2})^2 - \frac{7}{2} - 6 = 2\frac{3}{4}$	Replace x with $3\frac{1}{2}$
-2	$y = (-2)^2 - (-2) - 6 = 0$	Replace xy with -2

Thus, the curves intersect at the points $(3\frac{1}{2},2\frac{3}{4})$ and $(-2,0)$. This is illustrated in the figure.

3. Find the points of intersection of the hyperbolas $x^2 - 2y^2 = 4$ and $8y^2 - 2x^2 = 1$.

$x^2 = 2y^2 + 4$ — Solve the first equation for x^2

$8y^2 - 2(2y^2 + 4) = 1$ — Replace x^2 with $2y^2 + 4$ in the second equation

$y = \pm\frac{3}{2}$ — Solve for y

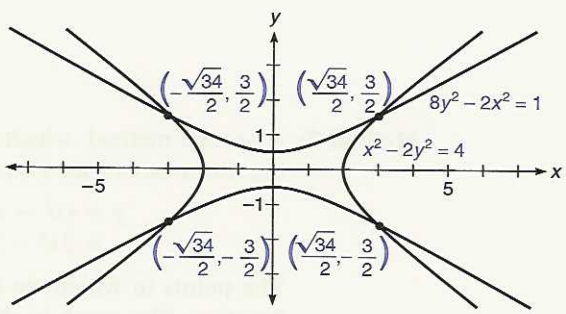
Now find x.

$x^2 = 2y^2 + 4$ — First equation

$x^2 = 2(\frac{9}{4}) + 4$ — $y^2 = \frac{9}{4}$

$x = \pm\sqrt{\frac{17}{2}} = \pm\frac{\sqrt{34}}{2}$ — $\sqrt{\frac{17}{2}} = \frac{\sqrt{17}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{34}}{2}$

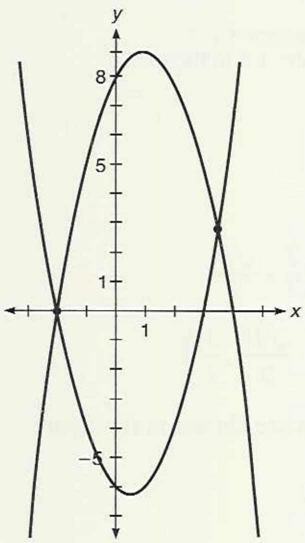
Thus, we obtain four solutions: $\left(\frac{\sqrt{34}}{2}, \frac{3}{2}\right)$, $\left(-\frac{\sqrt{34}}{2}, \frac{3}{2}\right)$, $\left(\frac{\sqrt{34}}{2}, -\frac{3}{2}\right)$, $\left(-\frac{\sqrt{34}}{2}, -\frac{3}{2}\right)$. These solutions are shown in the figure.



Graphing calculators can be conveniently used to obtain approximate solutions to systems in which both equations represent functions. In this situation, both equations are solved for y, or can be easily solved for y. Thus parts 1 and 2 of example 11–4 A are suitable for a graphing calculator. Example 11–4 B illustrates how to use the calculator in this case.

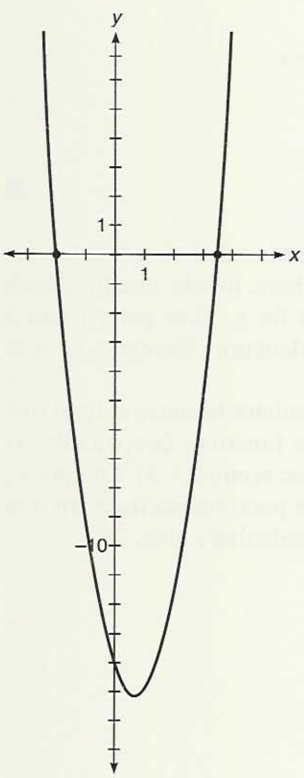
Part 3 cannot be easily done on a graphing calculator because at least one of the equations represents a relation that is not a function. Graphically, at least one of the graphs fails the vertical line test (see section 3–5). Of course, we saw how to deal with hyperbolas and ellipses in previous sections, so it is possible to attack these problems with a graphing calculator also.

■ ***Example 11–4 B*** Solve the system of equations $y = x^2 - x - 6$ and $y = -x^2 + 2x + 8$ by obtaining approximate solutions with a graphing calculator.



We will illustrate two methods for solving this problem.

Method 1: Graph both equations in the same coordinate system. The point(s) of intersection of the graphs are the solutions. In the figure, there are two such points. In example 11–4 A, these were shown to be (3.5,2.75) and (−2,0). Use the TRACE and ZOOM features to estimate values near these actual values.



Method 2: A second method, which will obtain just the x value, is to graph the difference of the two x-expressions, in this case,

$$\begin{aligned} y &= (x^2 - x - 6) - (-x^2 + 2x + 8) \\ &= 2x^2 - 3x - 14 \end{aligned}$$

The points in which we are interested are the zeros of this new function. The graph is shown in the figure, where we can accurately estimate the x values to be -2 and 3.5. The y values can be found by evaluating either of the two equations

$$y = x^2 - x - 6 \quad \text{and} \quad y = -x^2 + 2x + 8$$

for y using $x = -2$ and $x = 3.5$.

Note In this case $0 = 2x^2 - 3x - 14$ can be accurately solved by factoring or the quadratic formula. We can also use the program NEWTON presented in section 4–3. ■

Nonlinear inequalities

Section 10–4 discussed linear inequalities. An example is $2x + y < 8$. The solution to such an inequality is a half-plane bounded by the corresponding equality $2x + y = 8$.

The solution to nonlinear inequalities is also a part of the plane, although not a half-plane. The method for finding this part of the plane is similar to that for linear inequalities.

To solve a nonlinear inequality

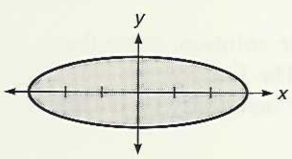
- Graph the corresponding equality. This divides the plane up into two or more regions.
- Try a test point from each region in the original inequality to determine which regions form the solution set.

Note There can be more than two regions formed by a nonlinear inequality. All regions should be checked.

Example 11–4 C illustrates graphing nonlinear inequalities.

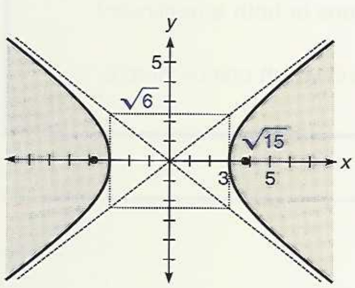
■ *Example 11–4 C*

Graph the solution to the nonlinear inequality.



1. $\frac{x^2}{9} + y^2 \leq 1$

Graph the corresponding equality $\frac{x^2}{9} + y^2 = 1$. This is an ellipse centered at the origin. The x-intercepts are $(\pm3,0)$ and the y-intercepts are $(0,\pm1)$. The ellipse divides the plane into the region inside and outside of it. The test point of (0,0) is true since $\frac{0^2}{9} + 0^2 \leq 1$ is true. Thus, the part of the plane that contains the origin is part of the solution set. Also, this is a weak inequality, so the ellipse itself is part of the solution set. Indicate this by drawing the ellipse with a solid line.



2. $\frac{x^2}{9} - \frac{y^2}{6} \geq 1$

The graph of $\frac{x^2}{9} - \frac{y^2}{6} = 1$ is part 1 of example 11–3 A. We take test points in each of the *three* regions formed by the hyperbola: $(-5,0)$, $(0,0)$, and $(5,0)$. The points $(-5,0)$ and $(5,0)$ satisfy the original inequality, but the test point $(0,0)$ does not. The solution is therefore the two regions that do not include the origin. ■

Systems of nonlinear inequalities

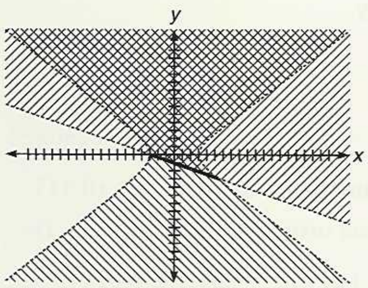
A system of nonlinear inequalities is a system of inequalities in which at least one of the inequalities is nonlinear. The solution set to such a system is the intersection of the solution set of each of the inequalities. Graphically this is where the solutions to the individual inequalities overlap. This is the same as for systems of linear inequalities (section 10–4). Example 11–4 D illustrates.

■ **Example 11–4 D** Graph the solution set to the system of nonlinear inequalities $\frac{x^2}{9} - \frac{y^2}{4} < 1$ and $3y + x \geq -3$.

We first graph the corresponding equalities.

$\frac{x^2}{9} - \frac{y^2}{4} = 1$ is a hyperbola with $a = 3$, $b = 2$.

$3y + x = -3$ is a straight line with y-intercept -1 and x-intercept -3. Now try test points to determine the parts of the plane that solve each individual inequality.



The solution to the first inequality is the part of the plane determined by the hyperbola that contains the point (0,0). The half-plane that contains (0,0) is part of the solution to the second inequality.

The edge of the hyperbola can *not* be part of the solution, since the hyperbola is described by a strong inequality (<). The line $3y + x = -3$ can form part of the solution because this is a weak inequality ($\geq$). This is shown by graphing with a solid line.

The solution is where the solution sets overlap. This is the cross-hatched portion of the graph along with the part of the straight line that is solid. ■

Mastery points

Can you

- Solve systems of two equations in which one or both is nonlinear?
- Solve a nonlinear inequality?
- Solve systems of two or more inequalities in which one or more is nonlinear?

Exercise 11–4

Solve the following systems of equations.

1. $y = 2x + 1$
$y = x^2 + x - 5$

2. $y = -x + 2$
$y = x^2 - 3x - 1$

3. $y = x^2 - x$
$y = 4 - x^2$

4. $y = x^2 - 9$
$y = 4 - x^2$

5. $y = 3x^2 - 2x - 4$
$y = x^2 + x + 1$

6. $y = x^2 - 3x - 4$
$y = 2 - x - 3x^2$

7. $y = x^2 - 3x - 4$
$y = x - 7$

8. $y = x^2 + x - 20$
$y + 3x + 2 = 0$

9. $y = x^2 + 3x - 8$
$y - x = 2$

10. $y = x^2 - 4x + 2$
$y + x = -6$

11. $4x^2 + y^2 = 4$
$y = 2x - 1$

12. $x^2 + 2y^2 = 2$
$y + x - 1 = 0$

13. $2x^2 + y^2 = 1$
$2x + y = 1$

14. $2x^2 + 3y^2 = 6$
$x = y + 2$

15. $2x^2 + y^2 = 3$
$2x - y = 1$

16. $\frac{x^2}{4} + y^2 = 1$
$x + y = 1$

17. $x^2 + \frac{y^2}{3} = 1$
$y = x - 2$

18. $y^2 + 2x^2 = 4$
$y = x - 1$

19. $x^2 - y^2 = 1$
$2y^2 - x^2 = 2$

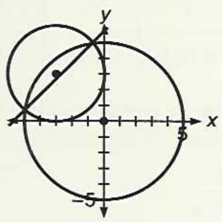
20. $3y^2 - x^2 = 6$
$y^2 - x^2 = 1$

21. $4y^2 - x^2 = 8$
$2x^2 - y^2 = 2$

22. $3y^2 - x^2 = 3$
$3x^2 - 2y^2 = 6$

23. A circle has center at (1,2) and is tangent to the line $y = \frac{1}{2}x - 3$. Find the equation of the circle. (*Hint:* Construct the radius which touches the line $y = \frac{1}{2}x - 3$ and find its equation. Find the point of intersection of these two lines, etc.) (See problem 25 for an alternate method of solving the problem.)

24. Two circles and a straight line will be engraved on a steel plate by a computer-controlled grinding machine. The circles are as shown in the figure, and the line passes through the two points where the circles intersect. Find the equation of the straight line.



25. A circle with center at (2,5) is tangent to the line $y = -x - 1$. Find the equation of the circle. (*Hint:* Write the equation of the circle, assuming its radius is some number r. Replace y in the equation of the circle by $-x - 1$ [since the circle and line touch at some point]. Solve for x. Because of the constraints in the problem there should only be one value of x in this solution. Find out how this could happen.)

Graph the solution to the following nonlinear inequalities.

26. $y > x^2 - 1$
27. $y < x^2 - 1$
28. $y < x^2 + 1$
29. $y \geq x^2 + 1$
30. $x^2 + y^2 \leq 9$
31. $x^2 + y^2 > 9$
32. $x^2 + y^2 > 16$
33. $x^2 + y^2 < 16$
34. $4x^2 + y^2 < 4$
35. $x^2 + \frac{y^2}{4} \geq 1$
36. $\frac{x^2}{4} + y^2 \geq 1$
37. $\frac{x^2}{4} + y^2 < 1$
38. $x^2 - \frac{y^2}{4} < 1$
39. $x^2 - \frac{y^2}{4} > 1$
40. $y^2 - \frac{x^2}{4} \geq 1$
41. $y^2 - \frac{x^2}{4} \leq 1$

Graph the solutions to the systems of inequalities.

42. $y > x^2 - 3$
$y \leq -x^2 + 4$

43. $y < x^2 - 1$
$y \leq x + 2$

44. $y < x^2 + 1$
$y > -2$

45. $y > x^2 + 2$
$2y < x + 6$

46. $x^2 + y^2 < 9$
$x + y > 0$

47. $x^2 + y^2 > 9$
$x + y > 0$

48. $x^2 + y^2 > 1$
$x^2 + y^2 \leq 16$

49. $x^2 + y^2 < 16$
$x^2 + y^2 < 4$

50. $x^2 + y^2 > 4$
$x^2 + y^2 < 1$

51. $x^2 + y^2 > 4$
$x^2 + y^2 > 9$

52. $x^2 + \frac{y^2}{4} < 1$
$x + \frac{y}{4} < \frac{1}{2}$

53. $x^2 + \frac{y^2}{4} > 1$
$y > x^2 - 4$

54. $\frac{x^2}{4} + y^2 \geq 1$
$y < -x^2 + 4$

55. $\frac{x^2}{4} + y^2 < 1$
$x - y > 2$

56. $x^2 - \frac{y^2}{4} < 1$
$2x \leq y + 2$

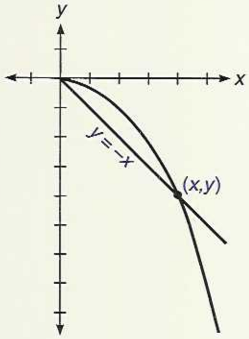
57. $x^2 - \frac{y^2}{4} > 1$
$\frac{x^2}{4} + y^2 < 1$

58. $y^2 - \frac{x^2}{4} > 1$
$x^2 + \frac{y^2}{4} < 1$

59. $y^2 - \frac{x^2}{4} < 1$
$x^2 + y^2 > 1$

60. In section 11–1 we saw that a falling object with horizontal velocity v (in ft/s) and no initial vertical velocity will follow the path $y = -\frac{16}{v^2}x^2$, where y is vertical distance in feet and x is horizontal distance in feet.

Is there a point in the fall of such an object where the vertical distance fallen equals the horizontal distance traveled? Note that at such a point the object would be on the path $y = -x$. See the figure.



61. If a rock is dropped into a well, it falls so that its distance s in feet is $s = 16t^2$, where time t is in seconds. Sound travels at about 1,100 feet per second. Thus the distance s in feet that sound travels in t seconds is $s = 1{,}100t$. If a rock is dropped into a well and the splash is heard after 3 seconds, how deep is the well? (*Hint:* Let z be the time it takes the rock to fall and hit the water. Let $3 - z$ be the time it takes for the sound to come back.)

62. A series of numbers $u_1, u_2, u_3, \ldots$ is constructed so that $u_{n+1} = \frac{1}{1 + u_n}$, $u_1 = 1$. For example, $u_2 = \frac{1}{1 + u_1} = \frac{1}{1 + 0} = \frac{1}{2}$. For another example, $u_3 = \frac{1}{1 + u_2} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$. Continuing in this fashion gives the series of numbers shown in the table.

u_1	u_2	u_3	u_4	u_5	u_6	u_7
1	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{8}{13}$	$\frac{13}{21}$
1	0.5	0.667	0.60	0.625	0.616	0.619

This series of fractions approaches a certain value, which is equivalent to solving the system of equations

$$y = \frac{1}{1 + x}$$
$$y = x$$

Solve this system and find the real number that the series approaches.

63. A rectangular piece of cloth has length 5 yards (yd) and width 3 yards. The length is to be increased by x yards and the width decreased by y yards. It is desired to have the new area at least as large as the old area. The new area is $(5 + x)(3 - y)$ yd^2, and the old area is $5 \cdot 3 = 15$ yd^2. Thus we want equation [1] $(5 + x)(3 - y) \geq 15$. Under the assumption that $x > 0$ it can be shown that this is equivalent to equation [2] $y \leq \frac{3x}{x + 5}$.

a. Show that equation [1] can be transformed into equation [2].

b. Graph the solution set to equation [2], including the constraints $x, y > 0$. (*Hint:* Graphing rational functions was covered in section 4–4.)

Skill and review

Graph each relation.

1. $y - 3x = -9$
2. $y - 3x^2 = -9$
3. $y^2 - 3x^2 = 9$
4. $y^2 + 3x^2 = 9$
5. $3y^2 + 3x^2 = 9$
6. $y = 2x^5 + x^4 - 10x^3 - 5x^2 + 8x + 4$

Chapter 11 summary

- **A parabola** is the set of all points equidistant from a line and a point not on that line. The line is called the directrix, and the point is called the focus.

$$y = \frac{1}{4p}(x-h)^2 + k$$

is the equation of a parabola with vertex at (h,k), focus at $(h, k+p)$, and directrix the line $y = k - p$.

- **An ellipse** is the set of all points such that the sum of the distances from each point to two other points, the foci, is constant. The line segment along the axis on which the foci lie is called the major axis; the other is the minor axis.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is the equation of an ellipse with center (h,k).

- **A hyperbola** is the set of all points such that the absolute value of the difference of the distances from each point on the hyperbola to two other points, the foci, is constant. The general equation of a hyperbola takes two forms.

[1] $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ Opens horizontally

[2] $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ Opens vertically

Chapter 11 review

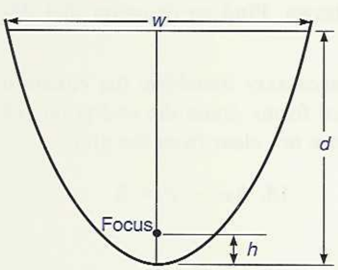
[11–1] Graph each parabola; state the x- and y-intercepts, focus, directrix, and vertex.

1. $y = -\frac{1}{8}x^2$ **2.** $y = x^2 - 6x + 5$
3. $y = -3x^2 - 4x + 4$ **4.** $y = x^2 + 4x + 6$
5. $x = y^2 - 7y - 8$ **6.** $x = y^2 - 4y + 4$

Determine the equation of a parabola with the given properties.

7. focus: $(1,-3)$, directrix: $y = 2$
8. focus: $(-3,-1)$, directrix: $y = -2$
9. vertex: $(2,-1)$, directrix: $y = -\frac{3}{4}$
10. focus: $(-4,2)$, vertex: $(-4,1)$
11. vertex: $(3,-1)$, x-intercepts: $2\frac{1}{2}, 3\frac{1}{2}$

Refer to the figure for the following problems.



12. $h = 8$, $d = 20$; find w **13.** $h = 6$, $w = 30$; find d
14. $w = 25$, $d = 25$; find h

[11–2] Convert each equation into the standard form for an ellipse if necessary. Then graph the equation. State the coordinates of the foci and end points of the major and minor axes where not clear from the graph.

15. $\frac{x^2}{4} + y^2 = 1$ **16.** $\frac{(x+3)^2}{16} + \frac{(y+1)^2}{25} = 1$
17. $12x^2 + 6y^2 = 24$ **18.** $4x^2 + 8y^2 = 4$
19. $x^2 - 6x + 4y^2 + 16y + 9 = 0$

Find the equation of the ellipse with the required properties.

20. foci: $(-3,0)$ and $(3,0)$; one x-intercept at 4
21. x-intercepts: $(\pm 3,0)$; y-intercepts: $(0,\pm 4)$

22. Two tacks are put in a board 6 inches apart and a string tied in a loop with length 8 inches is looped around the tacks. The ellipse is drawn. Find an equation that describes the ellipse.

[11–3] Graph each hyperbola. If necessary transform the equation into one of the two standard forms. Show the slant asymptotes and foci. State the coordinates of the foci and endpoints of the major axis where not clear from the graph.

23. $\frac{y^2}{20} - \frac{x^2}{5} = 1$ **24.** $x^2 - \frac{y^2}{2} = 1$
25. $4y^2 - x^2 = 4$ **26.** $8x^2 - 3y^2 = 8$
27. $\frac{(x-1)^2}{16} - \frac{(y+3)^2}{8} = 1$
28. $y^2 + 8y - 3x^2 + 12x + 1 = 0$

Categorize the graph of each equation as a point, line, circle, parabola, ellipse, or hyperbola (or no graph). Put each equation in the standard form for whichever geometric figure it represents.

29. $x^2 + 2x - 2y^2 + 8y = 16$
30. $9y^2 + 4x^2 - 36 = 0$
31. $2x^2 + 12x + y^2 - 6y + 23 = 0$
32. $4x^2 - 12x + 4y^2 - 30 = 0$
33. $y = x^2 + 3x - 4$
34. $x^2 + 8x + 4y^2 - 4y = 19$
35. $4x - 4y^2 + 20y - 23 = 0$

[11–4] Solve the following systems of nonlinear equations.

36. $y = x^2 - 3x + 4$, $y = \frac{2}{3}x + 4$

37. $\frac{x^2}{2} + y^2 = 1$, $y = -2x - 1$

38. $x^2 + 3y^2 - 8y - 1 = 0$, $x = y - 2$

39. $\frac{(x-1)^2}{12} - \frac{y^2}{16} = 1$, $x - 2y = 6$

40. $2y^2 - 6y - x^2 - 6x = 20$, $4x - 3 = y$

41. $x^2 + 5x - y^2 + 6y = 0$, $y = x - 2$

42. $y = -2x - 1$, $y = x^2 + x - 5$

43. $y = 3x^2 - 2x - 5$, $y = x^2 + 2x + 1$

44. A circle has center at $(-2,3)$ and is tangent to the line $y = 3x - 2$. Find the equation of the circle.

Graph the solutions to the nonlinear inequalities.

45. $y > x^2 - 6x - 8$
46. $4x^2 + y^2 \le 16$
47. $4x^2 - y^2 \ge 16$

Graph the solutions to the systems of nonlinear inequalities.

48. $y < -x^2 + 4$, $y > x - 2$

49. $x^2 + y^2 < 16$, $\frac{x^2}{4} + \frac{y^2}{16} > 1$

50. $\frac{y^2}{9} - x^2 > 1$, $\frac{x^2}{4} + \frac{y^2}{16} < 1$

Chapter 11 test

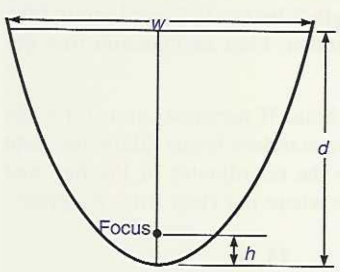
Graph each parabola; state the x- and y-intercepts, focus, directrix, and vertex.

1. $y = -4x^2$
2. $y = 2x^2 + 3x - 9$
3. $y = -x^2 - 2x + 8$
4. $x = y^2 - 4y - 12$

Determine the equation of a parabola with the given properties.

5. focus: (1,3), directrix: $y = 2$
6. vertex: $(-2,0)$, directrix: $y = 2$

Refer to the figure for problems 7 and 8.



7. $h = 2$, $d = 12$; find w
8. $w = 20$, $d = 10$; find h

Convert each equation into the standard form for an ellipse if necessary. Then graph the equation. State the end points of the major and minor axes and foci where not clear from the graph.

9. $4x^2 + y^2 = 16$
10. $4x^2 + y^2 - 10y = 11$
11. $2x^2 - 8x + 3y^2 + 9y = 15\frac{1}{4}$

12. Find the equation of an ellipse with x-intercepts at $(\pm 5,0)$ and y-intercepts at $(0,\pm 4)$.

13. Two tacks are put in a board 8 inches apart and a string tied in a loop with length 24 inches is looped around the tacks. The ellipse is drawn. Find an equation that describes the ellipse.

Graph each hyperbola. If necessary transform the equation into one of the two standard forms. State the end points of the major axis and foci where not clear from the graph.

14. $\frac{y^2}{9} - \frac{x^2}{4} = 1$
15. $4x^2 - y^2 = 8$
16. $\frac{(x-2)^2}{25} - \frac{(y+1)^2}{9} = 1$
17. $y^2 + 2y - 4x^2 + 16x = 19$

Categorize the graph of each equation as a point, line, circle, parabola, ellipse, or hyperbola (or no graph). Put each equation in the standard form for whichever geometric figure it represents.

18. $4x + 20y - 23 = 0$
19. $9y^2 - 3x^2 - 18 = 0$
20. $2x^2 + y^2 - 6y + 9 = 0$
21. $2x^2 + 2x - 2y^2 + 8y = 5$
22. $4x^2 - 12x + 4y^2 = 0$
23. $x^2 + 8x + y^2 - 4y = 20$
24. $x^2 + 3x - 6 - y = 0$

Solve the following systems of nonlinear equations.

25. $y = x^2 - 2x + 4$
$y = 2x + 1$

26. $\dfrac{x^2}{3} + y^2 = 1$
$y = x - 1$

27. $x^2 + 3y^2 - 8y - 2 = 0$
$y = x^2 - 2$

28. A circle has center at $(-1,3)$ and is tangent to the line $y = 2x - 2$. Find the equation of the circle.

Graph the solutions to the nonlinear inequalities.

29. $x^2 + 3y^2 > 9$

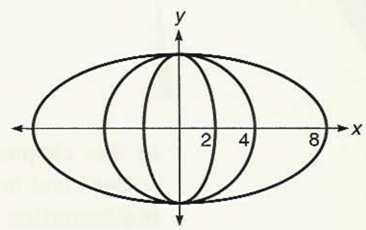
30. $3x^2 - 9y^2 \leq 27$

Graph the solutions to the systems of nonlinear inequalities.

31. $y > x^2 - 9$
$y < x + 2$

32. $x^2 + y^2 > 1$
$\dfrac{x^2}{4} + y^2 \leq 1$

33. An artist wants to construct two ellipses and a circle as shown. Find the equations of all three figures relative to an x-y coordinate system with center at the origin.



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